TITLE ON THE LATTICE AUTOMORPHISMS OF CERTAIN ALGEBRAIC GROUPS

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by

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Summary

In the first chapter we give an introduction, and a survey of known results, which we shall use throughout the dissertation.

In the second chapter we first prove that every projectivity of a connected reductive non-abelian algebraic group G over $K = \bar{F}_p$ is strictly index-preserving (Theorem 2.1.6.). Then we prove that every autoprojectivity of G induces an automorphism of the building canonically associated to G. Furthermore we show how certain autoprojectivities of G act on the Weyl group of G and on the Dynkin diagram of G.

In the third chapter we restrict our attention to simple algebraic groups over K. We prove that if G is a simple algebraic group over K of rank at least 2, then the problem whether every autoprojectivity of G is induced by an automorphism, is reduced to the problem whether every autoprojectivity of G fixing every parabolic subgroup of G is the identity. Namely, if we let

$$\Gamma(G) = \{\varphi \in \mathrm{Aut}\, \mathcal{L}(G) \mid P^{\varphi} = P \text{ for every parabolic subgroup } P \text{ of } G\},$$

we have

$$\mathrm{Aut}\, \mathcal{L}(G) = \Gamma \rtimes (\mathrm{Aut}\, G)^{*},$$

where $(\mathrm{Aut}\, G)^{*}$ is the group of all autoprojectivities of G induced by an automorphism (Theorem 3.4.9. and Corollary 3.4.15.).

In Chapter 4 we prove that actually $\Gamma = \{1\}$ if G has rank at least 3 and $p \neq 2$ (Theorem 4.6.5.), while in Chapter 5 we prove the same result, with different arguments, for the case of rank 1 (Corollary 5.2.6.) and 2, type A_2 excluded (Corollary 5.3.8.) (for groups of rank 1 we impose no restrictions on p).

Finally, in Chapter 6 we show that for the groups of type A_2 Theorem 4.6.5. does not hold. For this purpose we construct a non-trivial subgroup of the group $\Gamma(SL_3(\bar{F}_{23}))$ (Corollary 6.4.15.).

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Notation

If S , S_1 are sets , as usual

$S \subseteq S_1$ means that S is a subset of S_1 .

$S \cap S_1$ is the set theoretic intersection of S and S_1 .

$S \cup S_1$ is the set theoretic union of S and S_1 .

$S_1 \setminus S$ is the difference set of S_1 and S , namely the set of those elements belonging to S_1 but not to S .

$x \in S$ means that x belongs to S .

$\{x,y,z,\dots\}$ is the set consisting of the elements $x,y,z,\dots$.

$\mathbb{N}$ is the set of natural numbers . $\mathbb{N}_0$ is the set $\mathbb{N} \cup \{0\}$.

$\mathbb{Z}$ is the set of integers .

If m , $n \in \mathbb{Z}$, $m \geq n$ means that m is greater or equal n (in the natural order of $\mathbb{Z}$) , whereas $m > n$ means that m is strictly greater than n .

(m,n) is the greatest common divisor of m and n .

$m \mid n$ means that m divides n , while $m \nmid n$ means that m does not divide n .

$m \equiv n \bmod r$ means m , n congruent modulo r .

If G is a group ,

$H \leq G$ means that H is a subgroup of G .

$H < G$ means that H is a proper subgroup of G .

$H \lhd G$ means that H is a normal subgroup of G .

If $H \leq K \leq G$,

$\mathcal{C}(H)$ is the centraliser of H in G ,

$\mathcal{N}(H)$ is the normaliser of H in G ,

$\mathcal{C}_K(H)$ is the centraliser of H in K ,

$\mathcal{N}_K(H)$ is the normaliser of H in K ,

$Z(G)$ is the center of G .

If $S \subseteq G$, $<S>$ is the subgroup of G generated by the elements of S.

If x, y are in G, $[x,y]$ denotes the element $x^{-1}y^{-1}xy$.

If S, S_1 are subgroups of G, $[S,S_1] = < x^{-1}y^{-1}xy \mid x \in S, y \in S_1 >$.

$G' = [G,G]$ is the derived subgroup of G.

If $x, g \in G$ $x^g = g^{-1}xg$, ${}^gx = gxg^{-1}$.

If $x, y \in G$, $x \sim y$ means x, y conjugate in G, while $x \equiv y \mod H$ means $x^{-1}y \in H$.

If $g \in G$ $H^g = g^{-1}Hg$, ${}^gH = gHg^{-1}$.

If $L \leq G$, $H^L = < H^g \mid g \in L >$.

If $g \in G$, $|g|$ is the order of g ($|g| = \infty$ if $<g>$ is infinite cyclic).

If H_λ, for λ in a set A, are groups, we denote by $\underset{\lambda \in A}{\mathrm{Cr}}\, H_\lambda$ the cartesian product of the H_λ's, and by $\underset{\lambda \in A}{\mathrm{Dr}}\, H_\lambda$ the external direct product of the H_λ's.

If A is finite, $A = \{1,\dots,n\}$ say, we shall denote $\underset{\lambda \in A}{\mathrm{Dr}}\, H_\lambda$ by $H_1 \times \dots \times H_n$.

If $H_1,\dots,H_k$ are normal subgroups of G such that $<H_1,\dots,H_k>$ is the internal direct product of $H_1,\dots,H_k$, then we denote $<H_1,\dots,H_k>$ by $H_1 \times \dots \times H_k$. We shall usually denote by $\underset{\lambda \in A}{\oplus} H_\lambda$ the internal product of a family $(H_\lambda)_{\lambda \in A}$ of normal subgroups of G, if G is abelian.

If p is a prime, we say that an element x of G is a p-element if the order of x is a power of p. We say that x is a p'-element if the order of x is finite and coprime to p.

C_n denotes the (multiplicative) cyclic group of order n.

For every prime p, C_{p^∞} denotes the Prüfer group relative to p.

If G is a p-group (i.e. every element of G is a p-element),

$\Omega_i(G) = < x \mid x \in G$ and $x^{p^i} = 1 >$. We denote $\Omega_1(G)$ simply by $\Omega(G)$.

S_n is the symmetric group over n letters, A_n is the alternating group over n letters.

$\mathcal{L}(G)$ is the complete lattice of all the subgroups of G.
$[G/H]$ is the lattice of all subgroups of G containing H.

Φ is the set of roots, Π a fundamental system of Φ, Φ^+ a set of positive roots of Φ and Φ^- a set of negative roots of Φ.
For every root α, X_α is the root-subgroup corresponding to α, and x_α is a fixed algebraic isomorphism $x_\alpha : K \longrightarrow X_\alpha$.
We denote the diagonal matrix $\begin{pmatrix} a&0&0 \\ 0&b&0 \\ 0&0&c \end{pmatrix}$ by diag (a,b,c).

$N_{r,s}$ are the structure constants (r, s roots).
$M_{r,s,i}$ is the integer $(i!)^{-1} N_{r,s} N_{r,r+s} \cdots N_{r,(i-1)r+s}$ (r, s roots, i in $\mathbb{N}$).
If T is a subgroup of G, we denote by $\dot{w}$ any element of $\mathcal{N}(T)$ which is mapped to the element w of $\mathcal{N}(T)/T$ under the natural projection.

We shall usually write characters, cocharacters, the isomorphisms x_α and elements of the symmetric group on the left, while we shall usually write isomorphisms and projectivities of groups on the right.

Chapter 1 Introduction

The aim of the present dissertation is to study the group of lattice automorphisms of certain algebraic groups . As we shall deal both with properties of lattices and of algebraic groups , we give a short survey of the results of which we shall make use . We begin with lattices .

§ 1.1 A survey on lattices .

Let $(P,\leq)$ be a partially ordered set , and let X be any subset of P . We say that an element y of P is an *upper bound* of X if we have

$x \leq y$ for every x in X .

We say that an element a of X is a *least element* of X if we have

$a \leq x$ for every x in X (if it exists , it is then unique) .

Let us denote by $\mathcal{M}$ the set of all upper bounds of X . If $\mathcal{M}$ is not empty , and it has a least element , then this element is called *the least upper bound* of X , and it is denoted by *sup X* . Similarly one can define *the greatest lower bound* of X , which is denoted by *inf X* .

We can now give the definition of a lattice .

Definition : A *lattice* is a partially ordered set $(L,\leq)$ such that for every x , y in L , there exist both $\sup\{x,y\}$ and $\inf\{x,y\}$. L is said to be a *complete* lattice if each of its subsets X has $\sup X$ and $\inf X$.

It is clear that any non-empty complete lattice contains a least element , denoted by O, and a greatest element , denoted by I .

Let L be a lattice . Then , for every x , y in L , $\sup\{x,y\}$ will be denoted by $x \vee y$, and it will also be called the join of x and y ; $\inf\{x,y\}$ will be denoted by $x \wedge y$, and it will also be called the meet of x and y .

We also give the notion of sublattice and of cartesian product of lattices .

<u>Definition</u> : A *sublattice* of a lattice L is a subset X of L such that

$a \in X, b \in X \Longrightarrow a \vee b$ and $a \wedge b \in X$.

If we are given a family $(L_a)_{a \in A}$ of lattices, the *cartesianproduct* of the L_a's is the set $\{(x_a) \mid x_a \in L_a \ \forall a \in A\}$, and it is denoted by by $\underset{a \in A}{\mathrm{Cr}} L_a$. This becomes a lattice if we define the partial order componentwise.

<u>Examples</u> : Let X be any set. Then the set $\mathcal{P}(X)$ of all subsets of X partially ordered by inclusion is a complete lattice. For any family S of subsets of X, inf S is the set theoretic intersection of the elements of S, while sup S is the set theoretic union of the elements of S.

A partially ordered set C such that for every x, y in C we have

either $x \leq y$ or $y \leq x$,

is said to be totally ordered, and it is called a *chain*. Any chain is a lattice. The set $\mathbb{N}$ of the natural numbers with the usual total order is a chain, but it is not a complete lattice.

Let G be a group. We denote by $\mathcal{L}(G)$ the set of all subgroups of G partially ordered by inclusion. $\mathcal{L}(G)$ is a complete lattice. If $(X_a)_{a \in A}$ is a family of subgroups of G, then $\bigwedge_{a \in A} X_a$ is the set theoretic intersection of all the X_a's, while $\bigvee_{a \in A} X_a$ is the subgroup $\langle X_a \mid a \in A \rangle$ generated by all the X_a's.

As the union and the intersection of two normal subgroups of G are normal, the subset of all normal subgroups of G is a sublattice of $\mathcal{L}(G)$.

In any lattice L the following holds.

L 1	$x \wedge x = x$, $x \vee x = x$	(idempotent)
L 2	$x \wedge y = y \wedge x$, $x \vee y = y \vee x$	(commutative)
L 3	$(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$	(associative)
L 4	$x \wedge (x \vee y) = x \vee (x \wedge y) = x$	(absorption).

We have the following result :

Let L be a set with two binary operations $\vee$, $\wedge$ satisfying L 1–L 4 . Then L is a lattice , and conversely ([Bi] Theorem 8 page 10) .

If we are given $(L,\vee,\wedge)$ satisfying L 1–L 4 , then the partial order on L is given by the definition

$x \leq y \Leftrightarrow x \wedge y = x \ (\Leftrightarrow x \vee y = y)$

We shall write $(L,\leq)$ or $(L,\vee,\wedge)$ as it is more convenient .

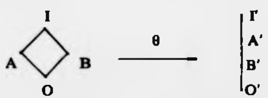
Let $(P,\leq)$, $(P',\leq')$ be partially ordered sets . A map $\theta : P \longrightarrow P'$ is said to be *order-preserving* if for every x , y in P such that $x \leq y$, we have $x^{\theta} \leq' y^{\theta}$.

Let $(L,\vee,\wedge)$, $(L',\vee',\wedge')$ be lattices . A map $\theta : L \longrightarrow L'$ is called a *lattice homomorphism* if we have

$$(x\wedge y)^{\theta} = x^{\theta} \wedge' y^{\theta} \quad \text{and} \quad (x\vee y)^{\theta} = x^{\theta} \vee' y^{\theta} \qquad \text{for every } x , y \text{ in } L .$$

θ will be called a *lattice isomorphism* if θ is bijective , and a *lattice automorphism* if $L = L'$, $\vee = \vee'$, $\wedge = \wedge'$, and θ is bijective .

If $\theta : L \longrightarrow L'$ is a lattice isomorphism , then both θ and θ^{-1} are order-preserving . We also have the converse , i.e. if we have a bijection θ between two lattices L and L' such that both θ and θ^{-1} are order-preserving , then θ is a lattice isomorphism . Note that the fact that θ is order-preserving is not enough to guarantee that θ is a lattice isomorphism , as the following example illustrates .



$I \mapsto I'$ $\quad A \mapsto A'$

$B \mapsto B'$ $\quad O \mapsto O'$

We now introduce the concept of modular and distributive lattices .

A lattice L is *modular* if it satisfies the "modular identity" (also called Dedekind Law)

$x \vee (y \wedge z) = (x \vee y) \wedge z$

for every x, y, z in L such that $x \leq z$.

Example : The lattice of normal subgroups of a group is a modular lattice.

A lattice L is *distributive* if we have

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for every x, y, z in L

(Note: this is equivalent to

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for every x, y, z in L.

(Theorem 9 page 11 in [Bi]).

Example: Any chain is a distributive lattice.

Before fixing our attention to the lattice $L(G)$ for a given group G, we give some other definitions.

Let P be a partially ordered set. Let a, b be elements of P. By "*a covers b*" it is meant that $a > b$, but $a > x > b$ for no x in P. If a, b are elements of P such that $a \geq b$, then the set $\{ x \in P \mid b \leq x \leq a \}$ is called an *interval* of P.

Let C be a finite chain with n elements. Then the *length* $\ell(C)$ of C is defined to be $n-1$. In general, the *length* $\ell(P)$ of a partially ordered set P is defined to be the least upper bound of the lengths of the chains in P. When $\ell(P)$ is finite, P is said to be of finite length. Any partially ordered set P of finite length is defined up to isomorphism by its covering relations, i.e. we have $a > b$ if and only if there exists a finite sequence $x_0, \ldots, x_n$ of elements of P such that $x_0 = a$, $x_n = b$ and x_{i-1} covers x_i for $i = 1, \ldots, n$.

If a partially ordered set P has the least element O, then an element a of P is said to be an *atom* if a covers O.

Let L be a lattice with O and I. Then if x is an element of L, by a *complement* of x in L is meant any element y of L such that $x \wedge y = O$ and $x \vee y = I$. L is called *complemented* if all its elements have complements.

<u>Example</u> : The modular lattice of all subspaces of a vector space over a field is complemented.

A *Boolean lattice* is a complemented distributive lattice. In a Boolean lattice complements are unique, and in any Boolean lattice of finite length, every element is a join of atoms (ex. 7a page 18 in [Bi]).

We shall now consider more closely how the lattice $L(G)$ of a given group G influences the structure of G.

§ 1.2 The lattice $L(G)$.

We first give some examples of the lattices $L(G)$ arising from some particular groups.

(1) Let G be a cyclic group C_{p^n} of order p^n, $n \geq 1$, p a prime. Then $L(G)$ is a chain of length n.

(2) Let G be the direct product of 2 copies of C_p, p a prime. Then $L(G)$ is represented by the diagram

So $L(G)$ is a modular lattice of length 2, and it has $p+1$ atoms.

(3) Let G be the symmetric group S_3. Then $L(G)$ is represented by the diagram

We now introduce some standard notation. Let G, $\bar{G}$ be groups. We shall call a *projectivity* of G onto $\bar{G}$ any lattice isomorphism

$$\varphi : L(G) \longrightarrow L(\bar{G}) .$$

We shall call an *autoprojectivity* of G any projectivity of G onto itself. We shall denote by $\mathrm{Aut}\, \mathcal{L}(G)$ the group of all autoprojectivities of G. Two groups $G, \bar{G}$ will be called *projective* if there exists a projectivity of G onto $\bar{G}$.

We shall use the usual abuse of notation $\varphi : G \longrightarrow \bar{G}$ to denote a projectivity φ of G onto $\bar{G}$, but we shall write $\varphi : \mathcal{L}(G) \longrightarrow \mathcal{L}(\bar{G})$ if there could be any ambiguity.

Many questions of different kinds arise in the attempt to relate the structure of a group to the structure of the lattice of its subgroups. For instance, let $\alpha : G \longrightarrow \bar{G}$ be an isomorphism of groups. Then we have of course $N \lhd G \Leftrightarrow N^{\alpha} \lhd \bar{G}$, $Z(G)^{\alpha} = Z(\bar{G})$, $(G')^{\alpha} = \bar{G}'$, G is nilpotent $\Leftrightarrow \bar{G}$ is nilpotent, and so on.

The situation for projectivities is very different. Let us consider the groups $G = C_3 \times C_3$ and $\bar{G} = S_3$. From the examples (2) and (3), it follows that G and $\bar{G}$ are projective. Let φ be a fixed projectivity of G onto $\bar{G}$. Let σ be an involution of $\bar{G}$, and let A be the subgroup of G such that $A^{\varphi} = \langle\sigma\rangle$. Then we have

$A \lhd G$ but $A^{\varphi} \ntriangleleft \bar{G}$,

$Z(G)^{\varphi} = \bar{G}$, while $Z(\bar{G}) = \{1\}$

$(G')^{\varphi} = \{1\}$ while $\bar{G}' = A_3$,

G is abelian, while $\bar{G}$ is even not nilpotent,

A has order 3, while A^{φ} has order 2.

Many results are nevertheless known. In the following we are going to recall those which will be used throughout the dissertation.

Let $\varphi : G \longrightarrow \bar{G}$ be a projectivity between the groups G and $\bar{G}$. Then

(i). G is finite if and only if $\bar{G}$ is finite.

(ii). G is cyclic if and only if $\bar{G}$ is cyclic (This result was obtained by Ore in 1938, while classifying the groups whose lattice of subgroups is distributive [O]).

(iii). $L(G)$ is a cartesian product of lattices L_a $(a \in A)$ if and only if G is a direct product of groups G_a $(a \in A)$ such that $L(G_a)$ is isomorphic to L_a for every a in A, and the order of any element of G_a is finite and coprime to the order of any element of G_b for every b in A, $b \neq a$ ([S] theorem 4 page 5).
(iv). G is solvable if and only if $\bar{G}$ is solvable (This was first obtained by Zappa in 1951 for the finite case [Za] . Yacovlev then proved the general result in 1970 [Y] . In his paper Yacovlev also gave a bound for the derived length of $\bar{G}$ in terms of a cubic polynomial in the derived length n of G . The best result in this direction was obtained by Busetto and Napolitani in 1988 , and puts the bound down to $3n-1$ [B,N]).
(v). If H, K are subgroups of G such that $K \leq H$ and $[H: K] < \infty$, then we have $[H^\varphi: K^\varphi] < \infty$ (This result was first proved by Rips [R] . Zacher gives a beautiful proof of this result in [Z_2]) .
(vi). In his book [S] , Suzuki defined φ to be *index-preserving* if given $H \leq K \leq G$, K cyclic and $[H: K] = n$, then we have $[H^\varphi: K^\varphi] = n$.
He also defined φ to be *strictly index-preserving* if given $H \leq K \leq G$ and $[H: K] = n$, then we have $[H^\varphi: K^\varphi] = n$.
An index-preserving projectivity maps p-groups to p-groups , so that every index-preserving projectivity is strictly index-preserving if G is finite . As a consequence of (v), it has been possible to prove this in general , i.e. a projectivity is strictly index-preserving if and only if it is index-preserving ([Z_2] Corollario 3).
(vii). If G is finite , and φ is not index-preserving , then G contains a normal Sylow complement N such that G/N is either a cyclic p-group , or an elementary abelian p-group ([S] Theorem 8 , page 45) .
(viii). If N is a normal subgroup of G such that N has no proper subgroup of finite index , then N^φ is normal in $\bar{G}$ ([Z_2] Corollario 2).

Let G , $\bar{G}$ be groups , and let $\alpha : G \longrightarrow \bar{G}$ be an isomorphism . Then we can define in a natural way the projectivity $\alpha^* : G \longrightarrow \bar{G}$ given by

$X^{\alpha^*} = X^\alpha$ for every $X \leq G$, where X^{α} is as usual the subgroup $\{ x^{\alpha} \mid x \in X \}$ of $\bar{G}$. α^* is called the projectivity induced by the isomorphism α. If $\bar{G} = G$, then α^* is called the autoprojectivity of G induced by the automorphism α. We therefore have a homomorphism $* : \mathrm{Aut}\, G \longrightarrow \mathrm{Aut}\, L(G)$ given by $\alpha \mapsto \alpha^*$ for every α in $\mathrm{Aut}\, G$.

An interesting question is whether a projectivity $\varphi : G \longrightarrow \bar{G}$ is induced by an isomorphism and, if so, if the isomorphism inducing φ is unique.
A group G is said to be *strongly lattice determined* if every projectivity $\varphi : G \longrightarrow \bar{G}$ of G onto a group $\bar{G}$ is induced by an isomorphism $\alpha : G \longrightarrow \bar{G}$. As for almost all the groups we are going to study φ will be induced by at most one automorphism, we shall use the convention to reserve always the term strongly lattice determined for groups such that every projectivity φ is induced by a unique isomorphism α.

It is clear that G is strongly lattice determined if and only if we have
G projective to $\bar{G} \Rightarrow G$ isomorphic to $\bar{G}$,
and the homomorphism $* : \mathrm{Aut}\, G \longrightarrow \mathrm{Aut}\, L(G)$ is bijective.

Known examples of finite groups which are strongly lattice determined are

the symmetric group S_n for $n \geq 4$ ([Z_1]);

the alternating group A_n for $n \geq 5$, $n \neq 3^r, 3^r+1$, where r is odd and greater than 2 [Sch];

the projective special linear groups $PSL_2(q)$ for $q = p^f \geq 4$, where p is a prime ([M_1]), and $PSL_3(3)$ ([M_2]), and the Suzuki groups $Sz(2^{n+1})$, where $n > 0$ ([M_2]).

For the simple groups A_n Schmidt gave also the structure of $\mathrm{Aut}\, L(A_n)$ for the exceptional values of n ([Sch]).

It is known that if G is a finite simple group, then we have

G projective to $\bar{G}$ $\Rightarrow$ G isomorphic to $\bar{G}$.

(The proof makes use of the classification of the finite simple groups. To our knowledge no direct proof is available at present). Therefore, in order to prove that a finite simple group is strongly lattice determined, it is enough to show that the homomorphism $*$: Aut G $\rightarrow$ Aut $L(G)$ is bijective. We also note that, by a result by Cooper ([Co], 2.2.2.), if G is a perfect group, then the map $*$ is injective. Hence a finite simple group G is strongly lattice determined if and only if the homomorphism $*$: Aut G $\rightarrow$ Aut $L(G)$ is surjective.

The groups studied by Metelli are a special family of finite simple groups of Lie type, and a conjecture was made that for all finite simple groups of Lie type the homomorphism $*$ was surjective. In 1985 Völklein proved this for the groups,

($\dagger$) $\quad B_\ell$, C_ℓ, $D_{2\ell}$, ${}^2D_{2\ell}$, ${}^3D_4$, E_7, E_8, F_4, G_2,

where the characteristic p of the base field is at least 11 ([V_1]).

From the theory of groups of Lie type, if G is a simple group of Lie type in the list ($\dagger$) over the finite field k, then G arises as the subgroup generated by the unipotent elements of the group of k-rational points of a certain adjoint simple algebraic group $\underline{G}$ defined over k. The groups in the list ($\dagger$) are exactly those arising from simple adjoint algebraic groups $\underline{G}$ whose Weyl groups have non-trivial center. Indeed, in 1985 Volklein ([V_1]) and the author ([Cs_1]) proved that for the groups $PSL_3(q)$ the homomorphism $*$ is not in general surjective. In 1988 we showed that q = 17 is the least prime-power number such that $PSL_3(q)$ has autoprojectivities not induced by automorphism ([Cs_2]).

This is in a certain sense the starting point for the present dissertation. The problem whether for the finite simple groups of Lie type G for which the Weyl group of $\underline{G}$ is trivial, the map $*$ is surjective, turned out to be much harder. Then it seemed an important step to consider the problem for simple algebraic groups over the algebraic closure of the finite fields F_q.

In the next paragraph we shall make a short survey of the properties of algebraic groups that we shall use .

§ 1.3 A survey on algebraic groups .

We shall deal only with algebraic groups over an algebraically closed field K . We recall that an *algebraic group* is a group G endowed with the structure of an algebraic variety such that the product map

$\mu : G \times G \longrightarrow G$ given by $\mu(x,y) = xy$ for every x , y in G

and the inversion map

$\iota : G \longrightarrow G$ given by $\iota(x) = x^{-1}$ for every x in G ,

are both algebraic maps .

We shall use the convention to reserve always the term algebraic group for those groups whose underlying variety is affine .

Examples : The *additive group* G_a is the affine line with the group law $\mu(x,y) = x+y$. We shall always denote this group by K . The *multiplicative group* G_m is the affine open subset K^* of the affine line , with group law $\mu(x,y) = xy$. We shall always denote this group by K^* .

If G is an algebraic group , then any closed subgroup of G is an algebraic group . If G_1 , G_2 are two algebraic groups then their direct product $G_1 \times G_2$ can be made into an algebraic group in a natural way .

The group $GL_n(K)$ of all non-singular $n \times n$ matrices over K may be regarded as an algebraic group as an affine open subset of K^{n^2} (hence as a closed subset of K^{n^2+1}) . Several other examples of algebraic groups can be obtained as closed subgroups of $GL_n(K)$, as the special linear group $SL_n(K)$, the symplectic group $Sp_n(K)$, the special orthogonal group $SO_n(K)$, and so on .

In fact every (affine) algebraic group over K can be viewed as a closed subgroup of $GL_n(K)$ for some n ([Hu] Theorem 8.6).

Let G , $\bar{G}$ be algebraic groups . A map $\alpha : G \longrightarrow \bar{G}$ ic called a *homomorphism of algebraic groups* if α is both an algebraic map and a homomorphism of groups. α is an isomorphism if it is bijective and both α and α^{-1} are homomorphisms of algebraic groups. We note that for a map α it is not enough to be a bijective algebraic homomorphism to be an isomorphism, as one can see from the following example.

Take $K = \bar{F}_p$, $G = K^*$, and $\alpha : G \to G$ given by $x \mapsto x^{p^n}$ for every x in K^*, where n is any natural number. Then α is a bijective algebraic homomorphism but not an algebraic isomorphism.

Let G be an algebraic group. Then G has a unique irreducible component (as a topological space) passing through the identity, which is denoted by G^0. G^0 is a normal closed subgroup of finite index of G. We also have that the cosets of G^0 are at the same time the connected components and the irreducible components of G ([Hu] Proposition 7.3).

An algebraic group G is said to be connected if $G^0 = G$. We recall that the dimension $\dim G$ of G is the dimension of the underlying variety of G.

Let $G, \bar{G}$ be algebraic groups, and let $\alpha : G \to \bar{G}$ be a homomorphism of algebraic groups. Then we have

$\ker \alpha$ is a closed subgroup of G;

$\operatorname{Im} \alpha$ is a closed subgroup of $\bar{G}$;

$\dim G = \dim \ker \alpha + \dim \operatorname{Im} \alpha$. ([Hu] Proposition 7.4B)

Also, if $(H_a)_{a \in A}$ is a family of closed and connected subgroups of G, then the group $< H_a \mid a \in A >$ they generate, is a closed and connected subgroup of G ([Hu] Proposition 7.5).

For any algebraic group G we can define the usual Jordan decomposition that we have for the groups $GL_n(K)$. In fact, for any algebraic group G, it is possible to give the definition of semisimple and of unipotent element, and then to show that for every x in G, there exist a unique semisimple element s and a unique unipotent element u in G such that $x = su = us$ ([Hu] Theorem 15.3). This fact is of key importance in the theory of algebraic groups .

We shall now shortly give the structure of abelian and solvable connected algebraic groups . Before this we give the definition of a torus and of a unipotent group .

A *torus* is an algebraic group T which is isomorphic to the direct product of a finite number , k say , of copies of K^* . Every element of a torus is semisimple , and the dimension of T is k .

An algebraic group will be called *unipotent* if each of its elements is unipotent .

Let G be an algebraic group . We denote by G_s the set of all semisimple elements of G , and by G_u the set of all unipotent elements of G . Suppose G is abelian . Then G_s and G_u are both closed subgroups of G , connected if G is connected , and we have $G = G_s \times G_u$ ([Hu] Theorem 15.5). Suppose now G is solvable and connected . Then G_u is a closed and connected normal subgroup of G . If T is any maximal torus of G , then we have
$G = G_u \rtimes T$ (a maximal torus of an algebraic group is by definition a closed subgroup of G which is a torus and which is not properly contained in any closed subgroup of G which is a torus). ([Hu] Theorem 19.3).

We now introduce the concept of semisimple and reductive algebraic group .

Let G be an algebraic group . The *radical* $R(G)$ of G is the largest closed connected normal solvable subgroup of G . The *unipotent radical* $R_u(G)$ of G is the largest closed connected normal unipotent subgroup of G (of course we have $R_u(G) = R(G)_u$).

<u>Definition</u> : Let G be a connected algebraic group . We say that G is *semisimple* if $G \neq \{1\}$ and $R(G) = \{1\}$. We say that G is *reductive* if

$G \neq \{1\}$ and $R_u(G) = \{1\}$.

Suppose G is a connected reductive group . Then the commutator subgroup G' of G is closed , connected , and semisimple if G is not abelian . If we denote by Z^0 the connected component of the center $Z(G)$ of G , then Z^0 is a torus , and we have a factorization $G = G'Z^0$ with $G' \wedge Z^0$ finite .

If G is a connected semisimple group , then G has a finite set of closed , connected , normal subgroups $G_1 ,...,G_k$ such that

(i) each G_i is a simple algebraic group ,

(ii) $[G_i , G_j] = \{1\}$ if $i \neq j$,

(iii) $G = G_1 \cdots G_k$,

(iv) $G_i \wedge (G_1 \cdots G_{i-1} G_{i+1} \cdots G_k)$ is finite for each i .

(a *simple algebraic group* is a non-commutative algebraic group which has no proper closed connected normal subgroups) ($[C_2]$ § 1.8).

The study of the structure of semisimple groups is therefore largely reduced to the study of simple algebraic groups . If G is a simple algebraic group , and we denote by Z its (finite) center , then G/Z is simple as an abstract group ([Hu] Corollary 29.5). In particular G is a perfect group (i.e. $G = G'$) . For the classification of simple algebraic groups , see $[C_2]$ § 1.11 .

We finally introduce the concept of a Borel subgroup of an algebraic group . Let G be an algebraic group . A *Borel subgroup* of G is a maximal closed connected solvable subgroup of G .

We summarise some of the properties of Borel subgroups .

All Borel subgroups are conjugate under G ([Hu] Theorem 21.3).

The maximal tori of G are those of the Borel subgroups of G , and they are all conjugate ([Hu] Corollary 21.3A) . The *rank* of an algebraic group is then defined to be the dimension of its maximal tori .

We have $\mathcal{N}(B) = B$ for every Borel subgroup of G ([Hu] Theorem 23.1).

If we denote by U the unipotent radical $R_u(B)$ (= B_u) of a Borel subgroup

B , then we have $\mathcal{N}(U) = B$ ([Hu] Corollary 23.1D) .

Let G be an algebraic group . A *parabolic subgroup* of G is any subgroup of G containing a Borel subgroup of G (parabolic subgroups are usually defined to be the closed subgroups containing a Borel subgoup , but it turns out , from the theory of groups with BN-pairs , as algebraic reductive groups are , that the closed subgroups containing a Borel subgroup are exactly all the subgroups containing a Borel subgroup . Then one proves that this is true for every algebraic group). We have $\mathcal{N}(P) = P$ for every parabolic subgroup of G ([Hu] Theorem 29.3(c)) .

Let G be a connected algebraic group , and let T be a maximal torus of G. Then the group $W(T) = \mathcal{N}(T)/\mathcal{C}(T)$ is finite ([C_2] § 1.9), and it is determined up to isomorphism by G , as any two maximal tori are conjugate . The abstract group W isomorphic to W(T) is called the Weyl group of G . If G is reductive , then we have $\mathcal{C}(T) = T$, and so $W(T) = \mathcal{N}(T)/T$. We shall always write W instead of W(T) .

Let us denote by X the group of all algebraic homomorphisms from T into K^* . Then X can be made into a group by defining

$(\chi_1+\chi_2)(t) = \chi_1(t)\chi_2(t)$ for every χ_1 , χ_2 in X , every t in T .

X is called the *character group* of T . If the dimension of T is k , then X is a free abelian group of rank k . Similarly we can define the *cocharacter group* Y of T to be the group of all algebraic homomorphisms of K^* into T . Y can be made into a group by defining

$(\gamma_1+\gamma_2)(\lambda) = \gamma_1(\lambda)\gamma_2(\lambda)$ for every γ_1 , γ_2 in Y , every λ in K^* .

Then Y is a free abelian group of rank k .

Let α be an algebraic endomorphism of K^*. Then there exists a unique n in $\mathbb{Z}$ such that α has the form $\lambda \mapsto \lambda^n$ for every λ in K^* . Hence we can define a map $X \times Y \to \mathbb{Z}$ by sending (χ,γ) to the unique integer $\langle\chi,\gamma\rangle$ such that $(\chi\gamma)(\lambda) = \lambda^{\langle\chi,\gamma\rangle}$ for every λ in K^* .

This map is non-degenerate , and gives rise to a duality between X and Y (i.e. $X \cong \mathrm{Hom}(Y,\mathbb{Z})$ and $Y \cong \mathrm{Hom}(X,\mathbb{Z})$). We make the Weyl group act both on X and on Y by defining

$${}^w\chi(t) = \chi(t^w) \qquad \text{and} \qquad \gamma^w(\lambda) = \gamma(\lambda)^w$$

for every χ in X , t in T , γ in Y and λ in K^* ($[C_2]$ § 1.9) .

Let G be a connected reductive group . Let T be a maximal torus of G and B be a Borel subgroup of G containing T . As B is solvable , it has the semidirect decomposition $B = UT$, where $U = B_u$. There exists a unique Borel subgroup B^- in G such that $B \wedge B^- = T$ ([Hu] Corollary 26.2C and the following exercise 6) . Also for B^- we have the semidirect decomposition $B^- = U^-T$, where $U^- = (B^-)_u$. U and U^- are closed and connected maximal unipotent subgroups of G normalised by T , and satisfying $U \wedge U^- = \{1\}$. We consider the minimal non-trivial closed connected subgroups of U and U^- which are normalised by T . These are all 1-dimensional , and isomorphic to K . We therefore have an action of T on the algebraic group K , and so each of our 1-dimensional unipotent groups determines an element of the character group X of T , as the group of algebraic automorphisms of K is isomorphic to K^* . The elements of X arising in this way are called the *roots* of G relative to T . The roots form a finite subset Φ of X . For every root α in Φ the 1-dimensional unipotent subgroup giving rise to α is denoted by X_α , and it is called a *root-subgroup* of G . For every α in Φ there exists an algebraic isomorphism $x_\alpha : K \longrightarrow X_\alpha$ such that we have

$$t x_\alpha(k) t^{-1} = x_\alpha(\alpha(t)k) \qquad \text{for every } k \text{ in } K .$$

([Hu] Theorem 26.3) . Whenever we consider root-subgroups X_α , we shall always assume already fixed the isomorphisms x_α .

Let Φ^+ be the set of roots arising from subgroups of U , and let Φ^- be the set of roots arising from subgroups of U^- . The roots in Φ^- are the negative of the roots in Φ^+ . Φ^+ is called the set of positive roots and Φ^- is called the set of negative roots . Let $\Pi = \{\alpha_1,...,\alpha_\ell\}$ be the set of positive roots which are not the sum of two positive roots . Π is called a *fundamental system* of Φ , and the elements of Π are called the *simple roots* . Each root α in Φ^+ is of the form $\alpha = n_1\alpha_1+...+n_\ell\alpha_\ell$, where each n_i is a non-negative integer . The *height* $ht(\alpha)$ of α is defined to be the integer $n_1+...+n_\ell$. Every root is the image of a simple root under the action of an element of W . Also , W contains a unique element w_0 such that $w_0(\Phi^+) = (\Phi^-)$. w_0 has order 2 .

There are several equivalent definitions of roots . It turns out that the set Φ is independent of the particular Borel subgroup B containing T , while choosing a Borel subgroup containing T is equivalent to choosing a fundamental system Π of Φ (or , equivalently , a set of positive roots Φ^+ of Φ) ([Hu] § 27.3) .

Every root α determines uniquely an element $\alpha^\vee$ of Y and an involution w_α of W . $\alpha^\vee$ is called the coroot corresponding to α, and w_α the reflection corresponding to α . We have $<\alpha,\alpha^\vee> = 2$ ($[C_2]$ § 1.9) .

Let now B be a Borel subgroup containing T , and let $\Pi = \{\alpha_1,...,\alpha_\ell\}$ be the fundamental system associated to B . The elements w_{α_i} of W will be called fundamental reflections , and will be denoted by s_i . W is generated as a Coxeter group by the set $\{s_1,...,s_\ell\}$, i.e. if m_{ij} is the order of s_is_j for $i \neq j$, we have

$$W = < s_1,...,s_\ell \mid s_i^2 = 1 \ \forall i , (s_is_j)^{m_{ij}} = 1 \ \text{ for } i \neq j > .$$

Let w be in W . Then a reduced expression for w is an expression

$$w = s_{i_1} \cdots s_{i_k} ,$$

whose length k is as small as possible . k will be called the *length* $\ell(w)$ of w . w_0 is the longest element of W .

We illustrate the above described situation by considering the group $SL_n(K)$. We may take T to be the subgroup of diagonal matrices in $SL_n(K)$. We may take B to be the subgroup of upper-triangular matrices . Then U is the subgroup of upper-unitriangular matrices. We also have for B^- the subgroup of lower-triangular matrices and for U^- the subgroup of lower-unitriangular matrices. The minimal proper subgoups of U normalized by T are the subgroups

$$X_{\alpha_{ij}} = \{ I + \lambda E_{ij} \mid \lambda \in K \}$$

(where E_{ij} is the elementary matrix with 1 in the (i,j) position and zero elsewhere) for $i < j$. The minimal proper subgoups of U^- normalized by T are the subgroups $X_{\alpha_{ij}}$ for $i > j$. The roots are the elements α_{ij} given by

$$\mathrm{diag}\,(\lambda_1,\dots,\lambda_n) \mapsto \lambda_i\lambda_j^{-1} \quad \text{for } i \neq j .$$

We have $\Phi^+ = \{ \alpha_{ij} \mid i<j \}$ and $\Phi^- = \{ \alpha_{ij} \mid i>j \}$. The fundamental system Π is the set $\{ \alpha_{12},\dots,\alpha_{n-1,n} \}$. For each root α_{ij} the corresponding coroot $\alpha_{ij}^\vee$ is the element given by

$$\lambda \mapsto \mathrm{diag}\,(1,\dots,1,\underset{i}{\lambda},1,\dots,\underset{j}{\lambda^{-1}},1,\dots,1) .$$

The normalizer N of T is the subgroup of monomial matrices in $SL_n(K)$. Thus the Weyl group W will be isomorphic to the symmetric group S_n. For each root α_{ij} the corresponding reflection $w_{\alpha_{ij}}$ is the permutation which transposes i,j and fixes the remaining symbols.

Let G be a semisimple group, and let $\Pi = \{\alpha_1,\dots,\alpha_\ell\}$ be a fundamental system of roots of G. We define the *Cartan integer* A_{ij} to be the integer $\langle\alpha_j,\alpha_i^\vee\rangle$ for every i,j. The matrix $A = (A_{ij})_{ij}$ is called the *Cartan matrix*. We have $A_{ii} = 2$ for every i, and $A_{ij} \in \{0,-1,-2,-3\}$ if $i \neq j$. If $A_{ij} \in \{-2,-3\}$, then $A_{ji} = -1$. Moreover we have

$$A_{ij} = 0 \iff A_{ji} = 0 .$$

Let us denote by n_{ij} the product $A_{ij}\,A_{ji}$. Then $n_{ij} \in \{0, 1, 2, 3\}$. n_{ij} is connected with the order m_{ij} of the element s_is_j of W. We have

$$n_{ij} = 0 \iff m_{ij} = 2 ,$$
$$n_{ij} = 1 \iff m_{ij} = 3 ,$$
$$n_{ij} = 2 \iff m_{ij} = 4 ,$$
$$n_{ij} = 3 \iff m_{ij} = 6 .$$

We define the *Dynkin diagram* of G . This is a graph with ℓ nodes , one for each simple root α_i . The nodes corresponding to the roots α_i , α_j for $i \neq j$ are joined by n_{ij} bonds . If $n_{ij} = 2$ or 3 then we place an arrow pointing from α_i to α_j if $A_{ji} \neq -1$ ($[C_2]$ § 1.11).

A connected semisimple group is simple if and only if its Dynkin diagram is connected . A simple group is determined by its Dynkin diagram and the location of the character group X between the *root lattice* $\mathbb{Z}\Phi$ and the *weight lattice* $\Omega = \mathrm{Hom}(\mathbb{Z}\Phi^{\vee}, \mathbb{Z})$. G is called *adjoint* if $X = \mathbb{Z}\Phi$, and *simply-connected* if $X = \Omega$ (i.e. if $Y = \mathbb{Z}\Phi^{\vee}$) . If G_{ad} and G_{sc} are the adjoint and the simply-connected simple groups with the same Dynkin diagram as G , then there exist surjective algebraic homomorphisms

$G_{sc} \longrightarrow G$ and $G \longrightarrow G_{ad}$.

The kernels of these homomorphisms are finite and central . The kernel of the latter map is equal to the center of G ($[C_2]$ § 1.11) .

If G is a semisimple group , and T is any maximal torus of G , then Z(G) coincides with the subgroup

$\{ t \in T \mid \alpha(t) = 1 \text{ for every } \alpha \text{ in } \Phi \}$,

of T , where Φ is the set of roots of G relative to T ([St] page 43) .

For the definition of diagonal , graph and field automorphisms of a semisimple group , we refer to [St] § 10 .

We finally recall the striking result that if G is a connected algebraic group , then G has a unique class of maximal unipotent subgroups , namely the unipotent radicals of the Borel subgroups of G ([Hu] Theorem 30.4(b) and the following remark) .

We shall now shortly summarise the results we are going to prove in this dissertation .

The main result of this thesis is that if p is an odd prime, then every autoprojectivity of a simple algebraic group G over $\bar{\mathbf{F}}_p$, not of type A_2, is induced by a unique automorphism of G. We prove this first for groups of rank at least 3 (Theorem 4.6.5.), and then we treat the case of rank 1 (here with no restrictions on p) and 2 (case A_2 excluded) with different arguments (Corollary 5.2.6., 5.3.8.).

In the last chapter we prove that in fact A_2 represents an exception, by exhibiting a family of autoprojectivities of the group $SL_3(\bar{F}_{23})$ which are not induced by any automorphism of $SL_3(\bar{F}_{23})$ (Corollary 6.4.15.).

Chapter 2 Lattice automorphisms of reductive algebraic groups

In this chapter we consider a connected reductive algebraic group over the field $\bar{\mathbb{F}}_p$ where p is any prime. We shall show that the group Aut $\mathcal{L}(G)$ acts in a natural way on the building $\Delta(G)$ canonically associated to G and we shall start studying this action.

We first give some notation that we shall use throughout the whole thesis. Let p be any prime. We shall always denote by K the algebraic closure of the field $\mathbb{F}_p$. Also, for every natural number n, we shall denote by K_n the unique subfield of K of order $p^{n!}$. Hence we have

$K_n \leq K_{n+1}$ for every n in $\mathbb{N}$, and

$\bigcup_{n\in\mathbb{N}} K_n = K$.

§ 2.1 Index-preserving projectivities.

In this paragraph we shall show that if G is a connected reductive non-commutative algebraic group over K and φ is a projectivity of G onto a group $\bar{G}$, then φ is strictly index-preserving.

For the purpose, we shall construct a family $(G_n)_{n\in\mathbb{N}}$ of finite subgroups of G such that $\bigcup_{n\in\mathbb{N}} G_n = G$ and each G_n has the property that every projectivity of G_n is strictly index-preserving.

Proposition 2.1.1. Let X be an affine variety defined over K. Then there exists a finite subfield F of K such that X is defined over F.

Proof: We have $X \subseteq K^m$ for some m in $\mathbb{N}$. The vanishing ideal $\mathcal{J}(X)$ of X in $K[X_1,\ldots,X_m]$ is finitely generated, by $f_1,\ldots,f_r$, say. But then there exists a finite subfield F of K such that $f_1,\ldots,f_r$ are all in $F[X_1,\ldots,X_m]$. Hence X is defined over F ([Hu] Proposition 34.1). #

We start now by supposing that G is a simple algebraic group over K. By the previous proposition, there exists a natural number $\bar{n}$ such that G is defined over $K_{\bar{n}}$. Then, for every $n \geq \bar{n}$, we can consider the finite subgroup $G(K_n)$ of the K_n-rational points of G. Also, we can take $\bar{n}$ large enough such that the derived group $G(K_n)'$ is perfect for every $n \geq \bar{n}$ ([T_1] Main theorem and Proposition 1.4).

We define $L_n = G(K_{\bar{n}+n-1})'$ for every n in $\mathbb{N}$.

<u>Proposition 2.1.2.</u> Let G be a simple algebraic group defined over K, and let $(L_n)_{n\in\mathbb{N}}$ be the family of subgroups of G defined above. Then we have the following:

L_n is a perfect finite subgroup of G for every n in $\mathbb{N}$;

$L_n \leq L_{n+1}$ for every n in $\mathbb{N}$;

$\bigcup_{n\in\mathbb{N}} L_n = G$.

<u>Proof</u>: From the definition of L_n, it follows that L_n is finite and perfect for every n in $\mathbb{N}$. Also, for every $n \geq \bar{n}$ we have $K_n \leq K_{n+1}$ and so $G(K_n) \leq G(K_{n+1})$. Hence we have

$L_n = G(K_{\bar{n}+n-1})' \leq G(K_{\bar{n}+n})' = L_{n+1}$ for every n in $\mathbb{N}$.

Finally let x be in G. As G is perfect, there exists elements $a_1,\dots,a_r,b_1,\dots,b_r$ in G such that $x = [a_1,b_1]\dots[a_r,b_r]$. From the fact that $\bigcup_{n\geq\bar{n}} K_n = K$, it follows that $\bigcup_{n\geq\bar{n}} G(K_n) = G$. Hence there exists M in $\mathbb{N}_0$ such that $a_1,\dots,a_r,b_1,\dots,b_r$ are all in $G(K_{\bar{n}+M})$. But then x lies in $G(K_{\bar{n}+(M+1)-1})' = L_{M+1}$, and we are done. #

Suppose now that G is a connected semisimple algebraic group over K. Then, from the structure of semisimple algebraic groups, we have:

$G = N_1 \cdots N_t$ where N_i is a simple algebraic group for every $i = 1,\dots,t$

$[N_r,N_s] = \{1\}$ for every r, s in $\{1,\dots,t\}$, $r \neq s$.

From Proposition 2.1.2. , for every $i = 1,\dots,t$ we have a family $(H_{i,j})_{j\in\mathbb{N}}$ of finite perfect subgroups of N_i , such that

$H_{i,j} \leq H_{i,j+1}$ for every j in $\mathbb{N}$, and

$\bigcup_{j\in\mathbb{N}} H_{i,j} = N_i$.

Let us denote by H_j the product $H_{1,j} \cdots H_{t,j}$, for every j in $\mathbb{N}$. Then H_j is a finite subgroup of G , as $[N_r,N_s] = \{1\}$ for every r , s in $\{1,\dots,t\}$, $r \neq s$. We have the following

Proposition 2.1.3. Let G be a connected semisimple algebraic group over K , and let $(H_j)_{j\in\mathbb{N}}$ be the family of subgroups of G defined above . Then

H_n is a perfect finite subgroup of G for every n in $\mathbb{N}$;

$H_n \leq H_{n+1}$ for every n in $\mathbb{N}$;

$\bigcup_{n\in\mathbb{N}} H_n = G$.

Proof : Clearly each H_j is finite and perfect , being the product of finite perfect groups. The remaining statements come immediately from Proposition 2.1.2. . #

We finally assume that G is a connected reductive non-commutative algebraic group over K . Then we have $G = T G'$, where T is the connected component of the centre $Z(G)$ of G , and the derived subgroup G' of G is semisimple . For G' we can then consider the family $(H_j)_{j\in\mathbb{N}}$ of Proposition 2.1.3. . On the other hand , T is a torus , and so it is isomorphic to the direct product of a finite number , k say , of copies of the multiplicative group K^* of K . Let n be any natural number coprime to p . Then K^* has a unique subgroup of order n . Call this subgroup D_n . For every j in $\mathbb{N}$, let $p^{\alpha_j} r_j$ be the cardinality of H_j , where $(p,r_j) = 1$. Let us denote by T_j the unique subgroup of T isomorphic to the subgroup $D_{r_j} \times \cdots \times D_{r_j}$ of $K^* \times \cdots \times K^*$ (k copies). Then T_j is a finite subgroup of $Z(G)$, and so the product $T_j H_j$ is a finite subgroup of G, for every j in $\mathbb{N}$. We denote this subgroup by G_j.

<u>Proposition 2.1.4.</u> Let G be a connected reductive non-commutative algebraic group over K, and let $(G_j)_{j \in \mathbb{N}}$ be the family of subgroups of G defined above. Then we have

$G_j \leq G_{j+1}$ for every n in $\mathbb{N}$;

$\bigcup_{j \in \mathbb{N}} G_j = G$.

<u>Proof</u>: As we have $H_j \leq H_{j+1}$, it follows that $r_j \mid r_{j+1}$, and so $T_j \leq T_{j+1}$ for every j in $\mathbb{N}$. Hence $G_j \leq G_{j+1}$ for every j in $\mathbb{N}$. To show that $\bigcup_{j \in \mathbb{N}} G_j = G$, we only need to show that $\bigcup_{j \in \mathbb{N}} G_j \geq T$, as, from Proposition 2.1.3. we have $\bigcup_{j \in \mathbb{N}} H_j = G'$. Now T is a torsion group, whose p-component is the identity. Hence we only need to show that for every natural number r coprime to p, there exists j in $\mathbb{N}$ such that $r \mid |H_j|$. For then we get $r \mid r_j$, and so if t is an element of order r of T then, by construction, t lies in T_j. For the purpose, we consider G'. We have $G' \neq \{1\}$, as G is non-commutative. Hence $G' = N_1 \cdots N_t$, where each N_i is a simple algebraic group. But then, if we take a maximal torus S_1 of N_1, we can find in S_1 an element s of order r. Therefore, with the notation used in the semisimple case, there exists j in $\mathbb{N}$ such that s lies in $H_{1,j}$. Hence we have s in H_j and so $r \mid |H_j|$, as we required. #

<u>Proposition 2.1.5.</u> Let j be any natural number, and let φ be any projectivity of the group G_j. Then φ is index-preserving.

<u>Proof</u>: Let j be in $\mathbb{N}$, and let φ be any projectivity of G_j. Suppose, for a contradiction, that φ is not index-preserving. Then there exists a Sylow r-subgroup R of G_j and a normal complement N of R in G_j, with R cyclic or elementary abelian ([S] Theorem 8 page 45). In particular the group G_j/N

is abelian, and so $N \geq G_j'$. But $G_j \geq H_j \Rightarrow G_j' \geq H_j' = H_j$, as H_j is perfect. Hence $N \geq H_j$, and so $r \nmid |H_j|$ as $r \nmid |N|$. Thus we must have $r \mid |T_j|$, as $G_j = T_j H_j$. But this is a contradiction, because every prime divisor of $|T_j|$ is also a divisor of $|H_j|$ by construction. Therefore φ is index-preserving. ∎

We can now state the following

Theorem 2.1.6. Let G be a connected reductive non-commutative algebraic group over K, and let φ be a projectivity of G onto a group $\bar{G}$. Then φ is strictly index-preserving.

Proof: As G is torsion free, it is enough to show that $|\langle g\rangle^{\varphi}| = |\langle g\rangle|$ for every g in G ([Z_2] Corollario 3). So let g be an element of G. By Proposition 2.1.4., there exists j in $\mathbb{N}$, such that g lies in G_j. If we consider the restriction φ_j from $L(G_j)$ to $L(G_j^{\varphi})$ of φ, we get, from Proposition 2.1.5., that φ_j is index-preserving. Hence we have

$$|\langle g\rangle^{\varphi}| = |\langle g\rangle^{\varphi_j}| = |\langle g\rangle| ,$$

and we are done. ∎

In [Z_2] Zacher also proved that a projectivity is strictly index-preserving if and only if it is index-preserving. For the rest of the thesis we shall therefore always write for convenience index-preserving to refer to a projectivity with the property of being strictly index-preserving.

Note: We excluded the case when G is an abelian connected reductive group, i.e. when G is a torus. We shall deal with this case in Chapter 5.

§ 2.2 The group Aut $\mathcal{L}(G)$ acts on the set of all Borel subgroups of G .

In this paragraph we consider a connected reductive non-commutative algebraic group G over K , and an autoprojectivity φ of G . We shall show that if B is a Borel subgroup of G , then B^{φ} is another Borel subgroup of G . It will follow that the group Aut $\mathcal{L}(G)$ of all autoprojectivities of G acts in a natural way on the building $\Delta(G)$ canonically associated to G .

First we recall that in our case an element x of G is unipotent (resp. semisimple) if and only if x has order a power of p (resp. x has order coprime to p) . From Theorem 2.1.6. we have the following

Proposition 2.2.1. Let G be a connected reductive non-commutative algebraic group over K , and let φ be an autoprojectivity of G . Let x , x_1 be elements of G such that $\langle x\rangle^{\varphi} = \langle x_1\rangle$. Then

x is unipotent (resp. semisimple) if and only if x_1 is unipotent (resp. semisimple) . ∎

Our aim is to show that if B is a Borel subgroup of G , then B^{φ} is a Borel subgroup , too . For the purpose we shall make use of the decomposition of B as the semidirect product of its unipotent radical and a maximal torus . We recall that in a reductive algebraic group there is a unique conjugacy class of maximal unipotent subgroups , which are precisely the unipotent radicals of the Borel subgroups of G ([Hu] Theorem 30.4b) .

Proposition 2.2.2. Let G be a connected reductive non-commutative algebraic group over K , and let φ be an autoprojectivity of G . Let U be a unipotent subgroup of G. Then U^{φ} is a unipotent subgroup of G . If U is maximal unipotent , then also U^{φ} is maximal unipotent .

Proof : Let U be a unipotent subgroup of G , and let x_1 be an element of U^{φ} . Then there exists x in U such that $\langle x\rangle^{\varphi} = \langle x_1\rangle$, because φ maps cyclic groups to cyclic groups . Then x is unipotent , and so is x_1 by Proposition 2.2.1. . Hence U^{φ} is a unipotent subgroup of G . Now suppose U is maximal unipotent , and let M be any unipotent subgroup of G containing U^{φ} . Then , by the first part of the proposition , $M^{\varphi^{-1}}$ is unipotent , and it contains U . But U is maximal unipotent , hence we have $M^{\varphi^{-1}} = U$, which gives $M = U^{\varphi}$. Therefore U^{φ} is a maximal unipotent subgroup of G . ∎

<u>Lemma 2.2.3.</u> Let G be a connected reductive non-commutative algebraic group over K , and let φ be an autoprojectivity of G . Let A be an abelian subgroup of G such that each element of A is semisimple and such that A has no proper subgroup of finite index . Then the closure $cl(A^{\varphi})$ of A^{φ} in G is a torus of G .

<u>Proof</u> : Let C be the closure of A^{φ} in G . We shall show that C is connected . As the connected component C° of C has finite index in C , it follows that $A^{\varphi} \wedge C^{\circ}$ has finite index , n say , in A^{φ} . We want to show that $n = 1$. Suppose on the contrary that $n > 1$. Then there exists a maximal subgroup M_1 of A^{φ} containing $A^{\varphi} \wedge C^{\circ}$. Hence $M = M_1^{\varphi^{-1}}$ is a maximal subgroup of A . But A is abelian , and so M must have finite index in A , which is a contradiction , as A has no proper subgroup of finite index by hypothesis . So we get $n = 1$, i. e. $A^{\varphi} \wedge C^{\circ} = A^{\varphi}$. Hence we have $A^{\varphi} \leq C^{\circ}$. But C° is closed , and so we get $C = cl(A^{\varphi}) \leq C^{\circ}$, which leaves us with $C = C^{\circ}$. Therefore C is connected . On the other hand , C is soluble because the projective image of a soluble group is soluble ([Y]) and the closure of a soluble subgroup is soluble ([B] Cor. 2 on page 110) . Hence , from the structure theorem of connected soluble algebraic groups , we get $C = C_u \rtimes T$, where C_u is the set of all unipotent elements of C , and T is a maximal torus of C . We have $A^{\varphi} \wedge C_u = \{1\}$, as A^{φ} consists only of semisimple elements by Proposition 2.2.1. . Hence we get

$$A^{\varphi} \cong A^{\varphi} C_u / C_u \leq T C_u / C_u \cong T \; ,$$

which implies that A^{φ} is abelian (Note : we could also get this result from the structure of locally finite modular groups) . Therefore C is itself abelian ,

as the closure of an abelian subgroup is abelian ([B] (2.1) d) . So , if we denote by C_s the set of all semisimple elements of C , from the structure theorem of abelian connected algebraic groups , we get $C = C_u \times C_s$, where C_s is a torus . We are now able to conclude by observing that , as the elements of A^φ are semisimple , we have $A^\varphi \leq C_s$, and so
$C = cl(A^\varphi) \leq C_s$. Hence $C = C_s$, and C is a torus . ∎

Proposition 2.2.4. Let G be a connected reductive non-commutative algebraic group over K , and let φ be an autoprojectivity of G . Let T be a maximal torus of G . Then T^φ is a maximal torus of G .

Proof : T is isomorphic to a direct product of copies of K^* , and so it is a divisible group , as K^* is a divisible group . Hence T has no proper subgroup of finite index . By Lemma 2.2.3. the closure of T^φ in G is a torus of G . Let us denote by T_1 this closure . The hypothesis of Lemma 2.2.3. holds for this group T_1 , and so $cl(T_1^{\varphi^{-1}})$ is a torus of G containing T . But T is a maximal torus of G , thus we have $cl(T_1^{\varphi^{-1}}) = T$. Also

$$T = cl(T_1^{\varphi^{-1}}) \geq T_1^{\varphi^{-1}} \geq T$$

gives $T_1^{\varphi^{-1}} = T$, so that $T_1 = T^\varphi$. Hence T^φ is closed in G , and it is a torus . Now suppose S is a torus of G containing T^φ . Then again $cl(S^{\varphi^{-1}})$ is a torus of G containing T , and so we must have $cl(S^{\varphi^{-1}}) = T$, as T is maximal . Hence we get

$$T = cl(S^{\varphi^{-1}}) \geq S^{\varphi^{-1}} \geq T ,$$

and therefore $T = S^{\varphi^{-1}}$. So $S = T^\varphi$, and T^φ is a maximal torus of G . ∎

We finally can consider the behaviour of Borel subgroups under autoprojectivities.

Proposition 2.2.5. Let G be a connected reductive non-commutative algebraic group over K , and let φ be an autoprojectivity of G . Let B be a Borel subgroup of G . Then B^φ is a Borel subgroup of G .

Proof : We have $B = UT$, where U is the unipotent radical of B , and T is any maximal torus of B . Then U is a maximal unipotent subgroup of G and T is a maximal torus of G ([Hu] Corollary A on page 135) . By Proposition 2.2.2. U^φ is a maximal unipotent subgroup of G and so it is the unipotent radical of a certain Borel subgroup of G . In particular U^φ is a closed and connected subgroup of G . On the other hand T^φ is a maximal torus of G by Proposition 2.2.4. , hence T^φ is closed and connected as well . Therefore the subgroup generated by U^φ and T^φ is closed and connected ([Hu] Proposition 7.5) . Thus $B^\varphi = U^\varphi \vee T^\varphi$ is a closed and connected subgroup of G . By definition , a Borel subgroup of G is a maximal closed connected soluble subgroup of G , and it is well known that a Borel subgroup of G is maximal even in the family of soluble subgroups of G ([Hu] Corollary A on page 143) . From this it follows that B^φ is a maximal soluble subgroup of G , as a projective image of a soluble group is soluble . In particular B^φ is a maximal soluble closed connected subgroup of G , and so it is a Borel subgroup of G . #

We conclude this paragraph with the following

<u>Theorem 2.2.6.</u> Let G be a connected reductive non-commutative algebraic group over K . Then the group Aut $\mathcal{L}(G)$ of all autoprojectivities of G acts in a natural way on the building $\Delta(G)$ canonically associated to G , in the sense that every autoprojectivity φ of G induces an automorphism of $\Delta(G)$.
<u>Proof</u> : We recall that the elements of $\Delta(G)$ are all the parabolic subgroups of G and that the partial order on $\Delta(G)$ is the reverse of the set-inclusion . Let now φ be an autoprojectivity of G . From Proposition 2.2.5., it is clear that φ induces a permutation on the set $\mathfrak{B}$ of all Borel subgroups of G . As the characteristic of φ is to preserve inclusions , it follows that φ permutes all the elements of $\Delta(G)$ and that it induces an automorphism of $\Delta(G)$. #

We get therefore a homomorphism from Aut $\mathcal{L}(G)$ to Aut $\Delta(G)$. Our aim is to study in detail this homomorphism .

§ 2.3 The action of Aut $\mathcal{L}(G)$ on the Weyl group of G .

In this paragraph we start studying the action of Aut $\mathcal{L}(G)$ on $\Delta(G)$. We shall fix a Borel subgroup B of G and a maximal torus T of B , and we shall restrict our investigation to the autoprojectivities of G fixing both B and T . We shall show that these autoprojectivities act on the Weyl group and on the Dynkin diagram of G .

We start with the following

Proposition 2.3.1. Let G be a connected reductive non-commutative algebraic group over K . Let T be any maximal torus of G and let φ be an autoprojectivity of G . Then we have

$\mathcal{N}(T)^{\varphi} = \mathcal{N}(T^{\varphi})$.

Proof : As T is a torus , it has no proper subgroup of finite index . Hence , from $T \triangleleft \mathcal{N}(T)$, it follows that $T^{\varphi} \triangleleft \mathcal{N}(T)^{\varphi}$ ([Z_2] Corollario 2) . Thus we have

(∗) $\mathcal{N}(T)^{\varphi} \leq \mathcal{N}(T^{\varphi})$.

Let $S = T^{\varphi}$. Then S is another maximal torus of G by Proposition 2.2.4. . We can apply (∗) to the couple (S , φ^{-1}) to get $\mathcal{N}(S)^{\varphi^{-1}} \leq \mathcal{N}(S^{\varphi^{-1}})$, which gives $\mathcal{N}(S) \leq \mathcal{N}(T)^{\varphi}$. Hence we obtain

$\mathcal{N}(S) \leq \mathcal{N}(T)^{\varphi} \leq \mathcal{N}(T^{\varphi}) = \mathcal{N}(S)$,

which leaves us with $\mathcal{N}(T)^{\varphi} = \mathcal{N}(T^{\varphi})$. #

From the previous proposition , given any maximal torus T of G and any autoprojectivity φ of G , we can define a projectivity

$\phi : \mathcal{N}(T)/T \longrightarrow \mathcal{N}(T^{\varphi})/T^{\varphi}$,

by the definition $(L/T)^{\phi} = L^{\varphi}/T^{\varphi}$, for every subgroup L such that $T \leq L \leq \mathcal{N}(T)$. We shall apply this to the case when $T^{\varphi} = T$, to obtain an autoprojectivity of $\mathcal{N}(T)/T$.

We fix now a Borel subgroup B of the connected reductive non-commutative algebraic group G over K and a maximal torus T of B.

Lemma 2.3.2. Let G be a connected reductive non-commutative algebraic group over K. Let φ be any autoprojectivity of G, and let B and T be as above. Then there exists an element g in G such that

$B^{\varphi} = B^{g}$ and $T^{\varphi} = T^{g}$.

Proof : From Proposition 2.2.5., B^{φ} is a Borel subgroup of G. As all Borel subgroups of G are conjugate in G, there exists an element x in G such that $B^{\varphi} = B^{x}$. Now $(T^{\varphi})^{x^{-1}}$ is a maximal torus of G contained in $(B^{\varphi})^{x^{-1}} = B$. So there exists an element b in B such that $(T^{\varphi})^{x^{-1}} = T^{b}$, as all maximal tori of B are conjugate in B. Let g be the element bx of G. Then we have

$$B^{\varphi} = B^{x} = B^{bx} = B^{g} \text{ and } T^{\varphi} = (T^{\varphi})^{x^{-1}x} = T^{bx} = T^{g},$$

and we are done. #

Therefore, given any autoprojectivity φ of G, we can adjust φ by an inner automorphism of G in order to fix both B and T.

We give the following

Definition 2.3.3. Let G be a connected reductive non-commutative algebraic group over K. For any couple (B,T), where B is a Borel subgroup of G and T is a maximal torus of B, we define $\Gamma_{B,T}$ to be the group of all autoprojectivities of G fixing B and T. #

We recall that once we fixed a maximal torus T of G, then we have the set Φ of the roots of G relative to T. Also, choosing a Borel subgroup B of G containing T, corresponds to choosing an order in Φ, so that we can define Φ^{+} to be the set of positive roots of Φ, and Φ^{-} to be the set of negative roots of Φ. We also denote by Π the set of simple roots. If we have

$\Pi = \{\alpha_1,...,\alpha_\ell\}$ then , for every root α in Φ , we have $\alpha = n_1\alpha_1+...+n_\ell\alpha_\ell$, where all n_i's are non-negative integers if α is in Φ^+, and all n_i's are non-positive integers if α is in Φ^- . Finally we denote by X_α the root-subgroup of G corresponding to the root α , for every α in Φ.

As we noted before , given an autoprojectivity φ fixing T , we can define an autoprojectivity $\bar{\varphi}$ of the Weyl group $W = \mathcal{N}(T)/T$ of G .

Proposition 2.3.4. Let φ be any autoprojectivity of G fixing T . Then the autoprojectivity $\bar{\varphi}$ of W induced by φ is index-preserving .

Proof : This follows from the fact that φ is strictly index-preserving , as we proved in Proposition 2.1.6. . #

At this point we recall that W is a Coxeter group , and we let

$$W = < s_1,...,s_\ell \mid s_i^2 = 1 \ \forall i , (s_is_j)^{m_{ij}} = 1 \ \text{ for } i \neq j >$$

be the presentation of W as a Coxeter group relative to the choice of the Borel subgroup B containing T . Then , if φ lies in $\Gamma_{B,T}$, from Proposition 2.3.4. , for every $i = 1,...,\ell$, there exists a unique involution $\bar{s}_i$ in W such that $<s_i>^{\bar{\varphi}} = <\bar{s}_i>$ for every $i = 1,...,\ell$, as $\bar{\varphi}$ maps cyclic groups to cyclic groups .

Proposition 2.3.5. Let i,j be in $\{1,...,\ell\}$, $i \neq j$. Then $|\bar{s}_i \bar{s}_j| = |s_i s_j|$ ($= m_{ij}$) .

Proof : Let us denote by W_{ij} the group generated by s_i and s_j , and by $\bar{m}_{ij}$ the order of $\bar{s}_i \bar{s}_j$. Then we have $W_{ij}^{\bar{\varphi}} = <s_i ,s_j>^{\bar{\varphi}} = <\bar{s}_i , \bar{s}_j>$. Hence both W_{ij} and $W_{ij}^{\bar{\varphi}}$ are dihedral groups , and they are of the same order by Proposition 2.3.4. . But we have

$$|W_{ij}| = 2m_{ij} \quad \text{and} \quad |W_{ij}^{\bar{\varphi}}| = 2\bar{m}_{ij} .$$

Hence we must have have $\bar{m}_{ij} = m_{ij}$ and we are done . #

We consider now the minimal parabolic subgroups of G containing B .

For every $i = 1,\dots,\ell$ let us fix a representative n_i in $\mathcal{N}(T)$ of s_i. Then the minimal parabolic subgroups of G containing B are $P_1,\dots,P_\ell$, where $P_i = \langle B, n_i \rangle$ for every $i = 1,\dots,\ell$. Also, for every $i = 1,\dots,\ell$ let $\bar{n}_i$ be an element of $\mathcal{N}(T)$ such that $\langle n_i\rangle^\varphi = \langle \bar{n}_i\rangle$. We therefore have

$$P_i^\varphi = \langle B, \bar{n}_i \rangle$$

for every $i = 1,\dots,\ell$. We have the following

Proposition 2.3.6. There exists a permutation σ of the set $\{1,\dots,\ell\}$ such that for every $i = 1,\dots,\ell$ we have $P_i^\varphi = P_{\sigma_i}$ and $\bar{s}_i = s_{\sigma_i}$.

Proof : Let i be an element of $\{1,\dots,\ell\}$. Let us denote by w_i the element $T\bar{n}_i$ of W, and let $w_i = s_{j_1}\cdots s_{j_k}$ be a reduced expression for w_i. Then if we denote by J the set $\{j_1,\dots,j_k\}$, we have $\langle B, \bar{n}_i \rangle = P_J$ ([C_2] Proposition 2.1.5.). But $\langle B, \bar{n}_i\rangle$ is a minimal parabolic subgroup of G containing B, hence J must consist of a single element, σ_i say. Hence we obtain

$$P_i^\varphi = \langle B, n_i \rangle^\varphi = \langle B, \bar{n}_i \rangle = P_J = P_{\sigma_i}.$$

Also, the previous reduced form of w_i must be $w_i = s_{\sigma_i}$. Therefore we get

$$Tn_{\sigma_i} = s_{\sigma_i} = w_i = T\bar{n}_i,$$

and so

$$\langle s_i\rangle^\phi = \langle Tn_i\rangle^\phi = \langle T, n_i\rangle^\varphi/T = \langle T, \bar{n}_i\rangle/T = \langle T\bar{n}_i\rangle = \langle Tn_{\sigma_i}\rangle =$$

$\langle s_{\sigma_i}\rangle$. But we had $\langle s_i\rangle^\phi = \langle \bar{s}_i\rangle$, which leaves us with $\bar{s}_i = s_{\sigma_i}$.

To show that the map $\sigma : \{1,\dots,\ell\} \to \{1,\dots,\ell\}$ is actually a bijection, we only need to observe that if we start with the inverse φ^{-1} of φ, then we can find in a similar way a map $\tau : \{1,\dots,\ell\} \to \{1,\dots,\ell\}$. From the fact that $\varphi\varphi^{-1} = \varphi^{-1}\varphi = 1$, it will then follow that $\sigma\tau = \tau\sigma = 1$, and so σ is bijective. #

We can now apply the previous results to get the following

Proposition 2.3.7. Let G be a connected reductive non-commutative algebraic group over K. Let B be a fixed Borel subgroup of G, and T be a maximal torus of B. Then every autoprojectivity of G fixing B and T induces a symmetry of the Dynkin diagram of G, if arrows are disregarded.

Proof : We recall that the Dynkin diagram of G is a graph with ℓ nodes, one for each simple root α_i. The nodes corresponding to the roots α_i, α_j with $i \neq j$ are joined by n_{ij} bonds, where n_{ij} depends only on the order m_{ij} of s_is_j. (The arrows depend on the Cartan integers A_{ij}, but here we are interested in the diagram without arrows). Let now φ be an autoprojectivity of G fixing B and T. Then, from Proposition 2.3.6., we have a permutation σ of $\{1,\dots,\ell\}$ such that $\bar{s}_i = s_{\sigma_i}$ for every $i = 1,\dots,\ell$, where $\bar{s}_i$ is the unique involution of $W = \mathcal{N}(T)/T$ such that $< s_i >^{\varphi} = < \bar{s}_i >$. Also, from Proposition 2.3.5., we get

$$m_{\sigma_i\sigma_j} = | s_{\sigma_i} s_{\sigma_j} | = | \bar{s}_i \bar{s}_j | = | s_i s_j | = m_{ij} .$$

In particular we have $n_{\sigma_i\sigma_j} = n_{ij}$, and so we obtain a symmetry of the Dynkin diagram of G if arrows are disregarded. #

We shall denote by $\bar{\sigma}$ the symmetry of the Dynkin diagram induced by φ. In particular we shall have $\bar{\sigma}(\alpha_i) = \alpha_{\sigma_i}$ for every $i = 1,\dots,\ell$.

Note : If instead of considering an autoprojectivity of a given connected reductive non-commutative algebraic group G over K, we consider a projectivity $\varphi : G \longrightarrow G_1$, where G, G_1 are connected reductive non-commutative algebraic groups over K, then with the same argument it is possible to show that if U, T, B are resp. a maximal unipotent subgroup, a maximal torus and a Borel subgroup of G, then U^{φ}, T^{φ} and B^{φ} are resp. a maximal unipotent subgroup, a maximal torus and a Borel subgroup of G_1. Also we still have $\mathcal{N}(T)^{\varphi} = \mathcal{N}(T^{\varphi})$ for every maximal torus T of G.

Now let B be a Borel subgroup of G, and T be a maximal torus of B. We denote by B_1 the Borel subgroup B^{φ} of G_1 and by T_1 the maximal torus T^{φ} of B_1. We can then define a projectivity $\bar{\varphi} : W \longrightarrow W_1$, where $W = \mathcal{N}(T)/T$ is the Weyl group of G and $W_1 = \mathcal{N}(T_1)/T_1$ is the Weyl group of G_1. Let

$$W = \langle s_1,\dots,s_\ell \mid s_i^2 = 1 \ \forall i, (s_is_j)^{m_{ij}} = 1 \ \text{for } i \neq j \rangle$$

be the presentation of W as a Coxeter group relative to the choice of the Borel subgroup B containing T, and

$$W_1 = \langle t_1,\dots,t_m \mid t_i^2 = 1 \ \forall i, (t_it_j)^{p_{ij}} = 1 \ \text{for } i \neq j \rangle$$

be the presentation of W_1 as a Coxeter group relative to the choice of the Borel subgroup B_1 containing T_1. Let $\bar{s}_i$ be the unique involution of W_1 such that $\langle s_i \rangle^\varphi = \langle \bar{s}_i \rangle$, for every $i = 1,\dots,\ell$. We shall have

(*) $|\bar{s}_i \bar{s}_j| = |s_i s_j|$ for every i, j in $\{1,\dots,\ell\}$, $i \neq j$.

Let $\mathcal{F}$ be the free group on the set $\{x_1, \dots, x_\ell\}$. We then define a homomorphism

$$\eta : \mathcal{F} \longrightarrow W_1$$

by extending the position $x_i \mapsto \bar{s}_i$ for every $i = 1,\dots,\ell$. Then η is surjective, as W is generated by the subgroups $\langle s_1 \rangle, \dots, \langle s_\ell \rangle$, and so W_1 is generated by the subgroups $\langle \bar{s}_1 \rangle, \dots, \langle \bar{s}_\ell \rangle$. Also we have that $\ker \eta$ contains

$$x_i^2 \ \forall i = 1,\dots,\ell, \ (x_ix_j)^{m_{ij}} \ \text{for } i \neq j$$

from (*). Hence we can define a surjective homomorphism $\bar{\eta} : W \longrightarrow W_1$ such that $\bar{\eta}(s_i) = \bar{s}_i$ for every $i = 1,\dots,\ell$. Similarly, starting from $\psi = \varphi^{-1}$, we can define a surjective homomorphism

$$\bar{\vartheta} : W_1 \longrightarrow W$$

such that $\bar{\vartheta}(t_i) = \bar{t}_i$ for every $i = 1,\dots,m$, where $\bar{t}_i$ is the unique involution of W such that $\langle t_i \rangle^\psi = \langle \bar{t}_i \rangle$, for every $i = 1,\dots,m$. Now, considering the minimal parabolic subgroups containing B and B_1, it is possible to define two maps

$$\sigma : \{1,\dots,\ell\} \longrightarrow \{1,\dots,m\} \quad \text{and} \quad \tau : \{1,\dots,m\} \longrightarrow \{1,\dots,\ell\},$$

such that $\bar{s}_i = t_{\sigma_i}$ for every $i = 1,\dots,\ell$ and $\bar{t}_j = s_{\tau_j}$ for every $j = 1,\dots,m$. It turns out that $\sigma\tau = \tau\sigma = 1$, and so $\ell = m$, and $\bar{\eta}$ and $\bar{\vartheta}$ are isomorphisms. In particular G and G_1 have isomorphic Weyl groups. Finally, we note that in the case $G = G_1$, we obtain a homomorphism from the group $\Gamma_{B,T}$ into the group $\mathrm{Aut}\,W$ of all automorphisms of W, given by the map $\varphi \mapsto \bar{\eta}$. ∎

§ 2.4 The action of the group $\Gamma_{B,T}$ on the Dynkin diagram of G.

Our aim is now to show that for every φ in $\Gamma_{B,T}$, there exists a graph automorphism δ of G which induces the same symmetry on the Dynkin diagram as φ does.

From now on we shall assume that G is a simple algebraic group over K. Then, for the existence of the graph automorphism δ we must consider separately the case when G has type B_2, F_4 or G_2. For this purpose we introduce, for every maximal torus T of G, the set $\mathfrak{B}_T$ of all Borel subgroups of G containing T. Then it is well known that

$$\mathfrak{B}_T = \{ B^{\dot{w}} \mid w \in W \},$$

where as usual we denote by $\dot{w}$ any representative of w in $\mathcal{N}(T)$ for every w in $W = \mathcal{N}(T)/T$, and B is any fixed Borel subgroup of G containing T. We also have that for any B in $\mathfrak{B}_T$, there exists a unique Borel subgroup B^- such that $B \wedge B^- = T$ ([Hu] Corollary 26.2C and exercise 6 on page 162). In fact we have $B^- = B^{\dot{w}_0}$, where $\dot{w}_0$ is the longest element of W. B^- is called the opposite of B with respect to T.

<u>Proposition 2.4.1.</u> Let G be a simple algebraic group over K. Let φ be an element of $\Gamma_{B,T}$. Then φ induces a permutation of the set $\mathfrak{B}_T$, and it fixes B^-. Moreover, if we denote by U, U^- resp. the unipotent radical of B and of B^-, then φ fixes both U and U^-.

<u>Proof</u>: From Proposition 2.2.5., we know that φ permutes the set of all Borel subgroups of G. But then, if B_1 lies in $\mathfrak{B}_T$, we shall have

$$B_1^{\varphi} \geq T^{\varphi} = T,$$

and so B_1^{φ} lies in $\mathfrak{B}_T$. Hence φ induces a permutation of $\mathfrak{B}_T$. Now we have

$$T = T^{\varphi} = (B \wedge B^-)^{\varphi} = B^{\varphi} \wedge (B^-)^{\varphi} = B \wedge (B^-)^{\varphi}.$$

Therefore we get $(B^-)^{\varphi} = B^-$ because of uniqueness of the opposite. Finally we have $U^{\varphi} = U$ and $(U^-)^{\varphi} = U^-$ because φ is index-preserving. ∎

We shall now show that every element φ of $\Gamma_{B,T}$ permutes the set of root-subgroups of G relative to T. We recall that the root-subgroups X_α are the minimal closed proper subgroups contained in U and U^- which are normalized by T ([C_2] page 18). It turns out that the X_α's are in fact the minimal proper subgroups contained in U and U^- which are normalized by T, because it is possible to show that every subgroup of U normalized by T must be closed and connected (and then the product of the X_α's it contains) ([Ch] Exp. 13, th. 1 d).

<u>Proposition 2.4.2.</u> Let G be a simple algebraic group over K. Let φ be in $\Gamma_{B,T}$, and let V be a unipotent subgroup of G such that T is contained in $\mathcal{N}(V)$. Then T is also contained in $\mathcal{N}(V^\varphi)$.

<u>Proof</u>: Let t be in T and let v be any element of V^φ. Then there exists v_1 in V and t_1 in T such that $\langle v_1\rangle^\varphi = \langle v\rangle$ and $\langle t_1\rangle^\varphi = \langle t\rangle$. By Proposition 2.1.4., there exists n in $\mathbb{N}$ such that v_1, t_1 are in G_n. We put $V_n = V \wedge G_n$. Then we have

$$V_n^{t_1} = (V \wedge G_n)^{t_1} = V^{t_1} \wedge G_n^{t_1} = V \wedge G_n = V_n,$$

so that $V_n \lhd \langle V_n, t_1\rangle$. But p does not divide the order of t_1, therefore V_n is the unique p-Sylow subgroup of $\langle V_n, t_1\rangle$. It follows that V_n^φ must be the unique p-Sylow subgroup of $\langle V_n, t_1\rangle^\varphi = \langle V_n^\varphi, t\rangle$, as φ is index-preserving. Therefore we get

$$V_n^\varphi \lhd \langle V_n^\varphi, t\rangle.$$

In particular we obtain that v^t lies in V_n^φ, as v lies in V_n^φ. So, for every v in V^φ, we proved that v^t lies in V^φ. Hence we have $(V^\varphi)^t \leq V^\varphi$ for every t in T. But then $(V^\varphi)^t = V^\varphi$ for every t in T, and so T is contained in $\mathcal{N}(V^\varphi)$. ∎

<u>Proposition 2.4.3.</u> Let G be a simple algebraic group over K. Let X_α be a root-subgroup of G (w.r.t. the choice of the maximal torus T). Then, for every φ in $\Gamma_{B,T}$, X_α^φ is a root-subgroup of G.

<u>Proof</u> : Let φ be in $\Gamma_{B,T}$. We have $T \leq \mathcal{N}(X_\alpha)$, and so , by Proposition 2.4.2. , we get $T \leq \mathcal{N}(X_\alpha{}^\varphi)$. Also , from $X_\alpha \leq U$ or $X_\alpha \leq U^-$, it follows by Proposition 2.4.1. , that $X_\alpha{}^\varphi \leq U$ or $X_\alpha{}^\varphi \leq U^-$. Now suppose V is a non-trivial subgroup of $X_\alpha{}^\varphi$ such that T is contained in $\mathcal{N}(V)$. Then we have $V^{\varphi^{-1}} \leq X_\alpha$ and , by Proposition 2.4.2. , $T \leq \mathcal{N}(V^{\varphi^{-1}})$. Hence we must have $V^{\varphi^{-1}} = X_\alpha$, as X_α is minimal with respect to the property of being contained in U or U^- and normalized by T . So we get $V = X_\alpha{}^\varphi$, and this means that $X_\alpha{}^\varphi$ is minimal with respect to the property of being contained in U or U^- and normalized by T . Hence $X_\alpha{}^\varphi$ is a root-subgroup of G . ▪

By the previous proposition , given an autoprojectivity φ in $\Gamma_{B,T}$, we can define a map

$\tau : \Phi \longrightarrow \Phi$ such that $X_\alpha{}^\varphi = X_{\tau(\alpha)}$

for every α in Φ .

Similarly , if we start with φ^{-1} , we can define a map $\rho : \Phi \longrightarrow \Phi$ such that $X_\alpha{}^{\varphi^{-1}} = X_{\rho(\alpha)}$, for every α in Φ . It turns out that $\tau\rho = \rho\tau = 1$, and so τ is a bijection . From Proposition 2.4.1. , we also have that $\tau(\Phi^+) = \Phi^+$ and $\tau(\Phi^-) = \Phi^-$.

We recall , from the previous paragraph , that for every φ in $\Gamma_{B,T}$ we have a permutation σ of the set $\{1,\dots,\ell\}$ such that $< s_i >^\varphi = < s_{\sigma_i} >$ for every $i = 1,\dots,\ell$, where $\{ s_1,\dots,s_\ell \}$ is the Coxeter generating set of the Weyl group $\mathcal{N}(T)/T$ with respect to the choice (B,T) . We also recall that , if for any $i = 1,\dots,\ell$ we choose a representative n_i of s_i in $\mathcal{N}(T)$, then the minimal parabolic subgroups containing B are $P_1 ,\dots, P_\ell$, where $P_i = \langle B,n_i \rangle$ for every $i = 1,\dots,\ell$, and that we have $P_i{}^\varphi = P_{\sigma_i}$ for every $i = 1,\dots,\ell$ (see Proposition 2.3.6.) .

We now introduce the thin chamber complex $\Sigma_\circ$ of all parabolic subgroups of G containing T . We shall take a closer look at this complex , and in general at the building $\Delta(G)$ in the next chapter .

Proposition 2.4.4. Let G be a simple algebraic group over K. Let φ be in $\Gamma_{B,T}$. Then we have

$$(B^{n_i})^{\varphi} = B^{n_{\sigma_i}} \qquad \text{for every } i = 1,\dots,\ell .$$

Proof : For every i in $\{1,\dots,\ell\}$, we have $P_i = \langle B , B^{n_i} \rangle$ ([C_2] Proposition 2.1.5.) . Now let us fix i in $\{1,\dots,\ell\}$. Then , from $B^{n_i} \le P_i$, we get

$$(B^{n_i})^{\varphi} \le P_i^{\varphi} = P_{\sigma_i} .$$

But $T \le B^{n_i} \Rightarrow T \le (B^{n_i})^{\varphi}$, and so $(B^{n_i})^{\varphi}$ lies in Σ_0, which is a thin chamber complex where the partial order is the reverse of the set theoretic inclusion . In particular P_{σ_i} contains only two Borel subgroups in Σ_0, namely B and $B^{n_{\sigma_i}}$. Hence we must have $(B^{n_i})^{\varphi} = B^{n_{\sigma_i}}$, as $B^{\varphi} = B$. #

Proposition 2.4.5. Let G be a simple algebraic group over K. Let φ be in $\Gamma_{B,T}$. Then we have

$$(X_{-\alpha_i})^{\varphi} = X_{-\alpha_{\sigma_i}} \quad \text{and} \quad X_{\alpha_i}{}^{\varphi} = X_{\alpha_{\sigma_i}}$$

for every $i = 1,\dots,\ell$.

Proof : For every $i = 1,\dots,\ell$, we have $X_{-\alpha_i} = U^- \wedge U^{n_i}$ ([C_1] page 101, 115 and [C_2] page 50 , 58) . From Proposition 2.4.4. , we have $(B^{n_i})^{\varphi} = B^{n_{\sigma_i}}$, and so we get $(U^{n_i})^{\varphi} = U^{n_{\sigma_i}}$, as U^{n_i} and $U^{n_{\sigma_i}}$ are respectively the unipotent radical of B^{n_i} and $B^{n_{\sigma_i}}$. Therefore we obtain

$$(X_{-\alpha_i})^{\varphi} = (U^- \wedge U^{n_i})^{\varphi} = U^- \wedge U^{n_{\sigma_i}} = X_{-\alpha_{\sigma_i}} .$$

We now consider the Borel subgroup B^-. We have $B^- = B^{n_0}$, where n_0 is any representative of w_0 in W. Then , as $B \wedge B^{n_0} = T$, we have $B^{n_i} \wedge B^{n_0 n_i} = T$ for every $i = 1,\dots,\ell$, and so $B^{n_0 n_i}$ is the opposite to B^{n_i} in Σ_0. Hence , for every $i = 1,\dots,\ell$, $(B^{n_0 n_i})^{\varphi}$ is opposite to $(B^{n_i})^{\varphi} = B^{n_{\sigma_i}}$ and it lies in Σ_0. Therefore we must have $(B^{n_0 n_i})^{\varphi} = B^{n_0 n_{\sigma_i}}$. So we obtain

$$X_{\alpha_i}{}^{\varphi} = (U \wedge U^{n_0 n_i})^{\varphi} = U \wedge U^{n_0 n_{\sigma_i}} = X_{\alpha_{\sigma_i}} .$$ #

Corollary 2.4.6. Let G be a simple algebraic group over K. Let φ be in $\Gamma_{B,T}$. Then , for every $i = 1,\dots,\ell$, we have

$\tau(\alpha_i) = \alpha_{\sigma_i}$ and $\tau(-\alpha_i) = -\alpha_{\sigma_i}$,

where τ and σ are the maps previously defined .

Proof : By the definition of τ we have $X_{\alpha_i}{}^{\varphi} = X_{\tau(\alpha_i)}$ and $X_{-\alpha_i}{}^{\varphi} = X_{\tau(-\alpha_i)}$. Hence , from Proposition 2.4.5. , it follows that $\tau(\alpha_i) = \alpha_{\sigma_i}$ and $\tau(-\alpha_i) = -\alpha_{\sigma_i}$. #

We can now consider the cases when G has type B_2 , F_4 or G_2 .

Lemma 2.4.7. Suppose X , Y are elementary abelian p-subgroups of a group G and let φ be an index-preserving autoprojectivity of G . Then we have

$[X,Y] = \{1\}$ if and only if $[X^{\varphi},Y^{\varphi}] = \{1\}$.

Proof : Let x , y be elements of X , Y respectively , and let $\bar{x}$, $\bar{y}$ be elements of X^{φ} , Y^{φ} respectively such that $\langle x\rangle^{\varphi} = \langle\bar{x}\rangle$ and $\langle y\rangle^{\varphi} = \langle\bar{y}\rangle$. Suppose $[X,Y] = \{1\}$. Then we have

$|\langle\bar{x}, \bar{y}\rangle| = |\langle x, y\rangle| \le p^2$,

as φ is index-preserving , and so $[\bar{x},\bar{y}] = 1$. It follows that $[X^{\varphi},Y^{\varphi}] = \{1\}$. Similarly $[X^{\varphi},Y^{\varphi}] = \{1\}$ implies that $[X,Y] = \{1\}$. Hence we have $[X,Y] = \{1\}$ if and only if $[X^{\varphi},Y^{\varphi}] = \{1\}$. #

Proposition 2.4.8. Let G be a simple algebraic group over K . Let φ be in $\Gamma_{B,T}$. Suppose the symmetry $\bar{\sigma}$ of the Dynkin diagram of G induced by φ is non-trivial . If G has type B_2 or F_4 , then the characteristic p of the field K must be 2 . If instead G has type G_2 , then we must have $p = 3$.

Proof : Assume first that G has type B_2 or F_4 . Then there exists two simple roots a , b interchanged by $\bar{\sigma}$, such that the positive roots which are linear combinations of a , b , are : a, b, $a+b$, $2a+b$ ([C_1] page 214) . Let us denote by $\lambda(a,b)$ this set of positive roots which are linear combinations of a , b . Assume now that the characteristic p is not 2 . We shall achieve a contradiction in a few steps . First we show that the bijection τ of Φ induced by φ fixes the set $\lambda(a,b)$. This is clear if G has type B_2 , for then we have $\lambda(a,b) = \Phi^+$. So suppose G has type F_4 . We have

$$[X_a, X_b] = X_{a+b}X_{2a+b}$$

by Proposition 2.5 and the following remark on page 66 in [B,T]. Hence we get $X_{a+b}X_{2a+b} \leq U_1$, where we denote by U_1 the subgroup $\langle X_a, X_b \rangle$ of U. Let Ψ be the set of roots γ in Φ such that X_γ is contained in U_1. Then we have $\lambda(a,b) = \Psi$, by 3.4 in [B,T], as $\lambda(a,b)$ is closed by construction, U_1 is closed and connected as it is the union of closed and connected subgroups, T is contained in $\mathcal{N}(U_1)$ as T is contained both in $\mathcal{N}(X_a)$ and $\mathcal{N}(X_b)$, and finally $T \wedge U_1 = \{1\}$ as $T \wedge U = \{1\}$ (for the definition of a closed set of roots see [B,T] 3.0).

Now let i, j be the elements of $\{1,\dots,\ell\}$ such that $a = \alpha_i$ and $b = \alpha_j$. By the definitions of σ and ϑ, we get $\sigma_i = j$, $\sigma_j = i$. Also, by Corollary 2.4.6., we have

$$\tau(a) = \tau(\alpha_i) = \alpha_{\sigma_i} = \alpha_j = b,$$

and similarly $\tau(b) = a$. Hence we obtain

$$U_1^{\varphi} = X_a^{\varphi} \vee X_b^{\varphi} = X_{\tau(a)} \vee X_{\tau(b)} = X_b \vee X_a = U_1.$$

Therefore, if γ is a root in Φ, we have

$$\gamma \in \lambda(a,b) \leftrightarrow X_\gamma \leq U_1 \leftrightarrow X_\gamma^{\varphi} \leq U_1^{\varphi} = U_1 \leftrightarrow X_{\tau(\gamma)} \leq U_1 \leftrightarrow$$
$$\tau(\gamma) \in \lambda(a,b).$$

Hence τ fixes $\lambda(a,b)$. In particular τ fixes the set $\kappa(a,b) = \lambda(a,b) \setminus \{a,b\}$.

For every root s in Φ, let us denote by $c(s)$ the number of roots r in $\kappa(a,b)$ such that $[X_s, X_r] = \{1\}$. We prove that $c(a) = c(b)$. Let $\gamma_1,\dots,\gamma_k$ be the roots r in $\kappa(a,b)$ such that $[X_a, X_r] = \{1\}$. Then, by Lemma 2.4.7., we have

$$[X_a^{\varphi}, X_{\gamma_i}^{\varphi}] = \{1\} \qquad \text{for } i = 1,\dots,k,$$

as for every root α of Φ the root-subgroup X_α is isomorphic to $(K,+)$, which is an abelian p-group. Hence we have

$$[X_b, X_{\tau(\gamma_i)}] = \{1\} \text{ for every } i = 1,\dots,k.$$

But the $\tau(\gamma_i)$'s are pairwise distinct, and so we have $c(b) \geq k = c(a)$, as τ fixes $\kappa(a,b)$. By symmetry we have $c(a) \geq c(b)$, so that $c(a) = c(b)$.

From Chevalley's Commutator Formula, we have

$[x_a(1), x_{a+b}(1)] = x_{2a+b}(\pm 2)$,

$[X_a, X_{2a+b}] = \{1\}$,

$[X_b, X_{a+b}] = \{1\}$,

$[X_b, X_{2a+b}] = \{1\}$.

Therefore we get $c(b) = 2$ and $c(a) = 1$, as $p \neq 2$. This is a contradiction , because we proved that we must have $c(a) = c(b)$. Hence if G has type B_2 or F_4 , the characteristic of the field is 2 .

Now assume that G has type G_2 . In this case σ interchanges the simple roots α_1 and α_2 . We choose the notation $(\alpha_1, \alpha_2) = (a,b)$ or $(\alpha_1, \alpha_2) = (b,a)$ in order to have $\Phi^+ = \{ a, b, a+b, 2a+b, 3a+b, 3a+2b \}$. As in the previous case , the bijection τ fixes the set $\lambda(a,b)\setminus\{a,b\} = \Phi^+\setminus\{a,b\}$, and again , if for every root s we denote by $c(s)$ the number of roots r in $\lambda(a,b)\setminus\{a,b\}$ such that $[X_s, X_r] = \{1\}$, we must have $c(a) = c(b)$. From Chevalley ' s Commutator formula , we have the following relations :

$[X_b, X_{a+b}] = \{1\}$,

$[X_b, X_{2a+b}] = \{1\}$,

$[X_b, X_{3a+2b}] = \{1\}$,

$[x_a(1), x_{a+b}(1)] = x_{2a+b}(\pm 2)x_{3a+b}(\pm 3)\, x_{3a+2b}(\pm 3)$,

$[x_a(1), x_{2a+b}(1)] = x_{3a+b}(\pm 3)$,

$[X_a, X_{3a+2b}] = \{1\}$,

$[X_a, X_{3a+b}] = \{1\}$.

Now suppose for a contradiction that $p \neq 3$. Then we have $c(b) \geq 3$, while $c(a) = 2$. Hence we do not have $c(a) = c(b)$, and this is a contradiction . Therefore we must have $p = 3$. #

Lemma 2.4.9. Let G be a simple algebraic group over K . Let φ be in $\Gamma_{B,T}$. Then there exists a graph automorphism δ of G such that we have

$$P^{\varphi} = P^{\delta}$$

for every parabolic subgroup P containing B .

Proof : Let σ be the symmetry of the Dynkin diagram of G induced by φ . We first prove the existence of a graph automorphism δ of G such that

$X_r{}^\delta = X_{\sigma(r)}$ for every simple root r of Φ. This follows immediately from Proposition 12.2.3. in [C_1] if all the roots in Φ have the same length. Therefore let us suppose that the roots are not all of the same length. Then, if $\sigma = 1$, the choice $\delta = 1$ will do. If instead we have $\sigma \neq 1$ then, by Proposition 2.4.8., we must have $p = 2$ if G has type B_2 or F_4, and $p = 3$ if G has type G_2. Then the existence of a graph automorphism δ such that $X_r{}^\delta = X_{\sigma(r)}$ for every simple root r of Φ, follows from 12.3.3. and 12.4.1. in [C_1]. To conclude, we need some properties of the graph automorphism δ. Following [C_1] § 12.2, 12.3, 12.4, we define a bijection $\vee$ of Φ, and we denote by $\bar{r}$ the image of r under $\vee$ for every r in Φ. We remind that $\vee$ comes from an isometry of the euclidean space spanned by the roots if all the roots have the same length, and it is defined by the map

$r = \Sigma\, n_i\alpha_i \mapsto \bar{r} = \Sigma\, n_i(\alpha_i,\alpha_i)(r,r)^{-1}\,\alpha_{\sigma_i}$ (sum over the set $\{1,\dots,\ell\}$),

in the case $\sigma \neq 1$ and G of type B_2, F_4 or G_2.

In any case it turns out that

$\vee(\Phi^+) = \Phi^+$, $\vee(\alpha_i) = \alpha_{\sigma_i}$ and $\vee(-\alpha_i) = -\alpha_{\sigma_i}$ for every $i = 1,\dots,\ell$

(see [C_1] 12.2.2., 12.3.2. and the following Observation). Also, from the construction of δ ([C_1] 12.2.3., 12.3.3., 12.4.1.) we have $X_r{}^\delta = X_{\bar{r}}$ for every root r of Φ. Hence we get $U^\delta = \prod X_r{}^\delta = \prod X_r = U$ (the product being over all the positive roots). We obtain

(*) $B^\delta = \mathcal{N}(U)^\delta = \mathcal{N}(U^\delta) = B$.

We also have

(**) $X_{-\alpha_i}{}^\delta = X_{\vee(-\alpha_i)} = X_{-\alpha_{\sigma_i}}$, for every $i = 1,\dots,\ell$.

We are now able to prove that $P^\varphi = P^\delta$ for every parabolic subgroup P containing B. We only need to show that this holds for the minimal parabolic subgroups $P_1, \dots, P_\ell$ of G containing B, as the interval $[G/B]$ is a Boolean lattice of finite length. From 2.6.4 in [C_2], we have $P_i = \langle B, X_{-\alpha_i}\rangle$ for every $i = 1,\dots,\ell$. Hence we obtain

$P_i{}^\varphi = \langle B^\varphi, X_{-\alpha_i}{}^\varphi\rangle = \langle B, X_{-\alpha_{\sigma_i}}\rangle$,

as $B^\varphi = B$, and $X_{-\alpha_i}{}^\varphi = X_{-\alpha_{\sigma_i}}$ by Proposition 2.4.5.. On the other hand we have

$$P_i^{\delta} = \langle B^{\delta}, X_{-\alpha_i}{}^{\delta} \rangle = \langle B, X_{-\alpha_{\sigma i}} \rangle ,$$

as $B^{\delta} = B$ by (*), and $X_{-\alpha_i}{}^{\delta} = X_{-\alpha_{\sigma i}}$ by (**). Therefore we have

$$P_i^{\varphi} = P_i^{\delta} \quad \text{for every } i = 1,\dots,\ell ,$$

and so $P^{\varphi} = P^{\delta}$ for every parabolic subgroup P containing B. ∎

From the previous lemma, given any autoprojectivity φ of G fixing a Borel subgroup B of G and a maximal torus T of B, we can adjust φ by a graph automorphism δ in order to fix T and all the parabolic subgroups of G containing B.

We now give the following

Definition 2.4.10. : Let G be a simple algebraic group over K. Then, for every Borel subgroup B of G and every maximal torus T of G contained in B, we define the group $\Gamma_{[G/B],T}$ to be the subgroup of all elements of $\Gamma_{B,T}$ fixing every parabolic subgroup of G containing B. ∎

Therefore, by Lemma 2.4.9., for every φ in $\Gamma_{B,T}$, there exists a graph automorphism δ of G such that $\varphi(\delta^{-1})^*$ lies in $\Gamma_{[G/B],T}$.

In the next chapter we shall study in detail the group $\Gamma_{[G/B],T}$.

Chapter 3 The action of Aut $L(G)$ on the Building $\Delta(G)$ associated to G.

In this chapter we shall consider in detail the action of the group Aut $L(G)$ on the Building $\Delta(G)$. First we recall some properties of $\Delta(G)$, and then we shall make use of these to determine the structure of Aut $L(G)$. For convenience we shall write Δ instead of $\Delta(G)$ when there is no ambiguity (we follow [T_2] Theorem 5.2) .

§ 3.1 The Building Δ .

The elements of Δ are all the parabolic subgroups of G , and the partial order on Δ is the reverse of the set-theoretic inclusion . Following the usual terminology we shall often call the elements of Δ faces . The chambers of Δ are the Borel subgroups of G . The set of apartments of Δ is the set

$\{ \Sigma_T \mid T \text{ is a maximal torus of } G \}$,

where , for every maximal torus T of G , Σ_T is the (finite) set of all parabolic subgroups of G containing T . More explicitly , if B is a Borel subgroup of G containing T , we have

$\Sigma_T = \{ {}^{w}P \mid P \geq B \text{ and } w \in W = \mathcal{N}(T)/T \}$ (recall that for every w in W we denote by w any representative of w in $\mathcal{N}(T)$) .

We make G act on Δ by left conjugation . If we define the normalizer of Σ_T in G by

$\mathcal{N}(\Sigma_T) = \{ g \in G \mid {}^{g}\Sigma_T = \Sigma_T \}$,

then we have $\mathcal{N}(\Sigma_T) = \mathcal{N}(T)$. From the existence of a unique conjugacy class of maximal tori in G , it follows that if T is a maximal torus of G and Σ is any apartment of Δ , then there exists g in G such that $\Sigma = {}^{g}\Sigma_T$.

Two faces X and Y of Δ have the same type if and only if X and Y are conjugate in G . Hence an automorphism ψ of Δ is type-preserving if and only if we have that X and X^{ψ} are conjugate in G for every X in Δ .

Finally we recall that if $\psi , \nu : \Sigma \longrightarrow \Sigma'$ are isomorphisms between two thin chamber complexes Σ, Σ' such that $X^{\psi} = X^{\nu}$ for every face X of a chamber of Σ, then we have $\psi = \nu$ ([T$_2$] Proposition 1.7).

We now fix a Borel subgroup B of G and a maximal torus T of B. The couple (B,T) will be fixed for the rest of this chapter. We shall denote by U the unipotent radical of B, by N the group $\mathcal{N}(T)$, by Σ_0 the apartment Σ_T, and we shall call it the fundamental apartment of Δ.

<u>Proposition 3.1.1.</u> Let g be an element of G. Let V_g be the set of all apartments of Δ containing the chamber ${}^{g}B$. Then the map $u \mapsto {}^{gu}\Sigma_0$, gives rise to a bijection of U onto V_g.

<u>Proof</u> : It is sufficient to consider the case when $g = 1$, for then the result follows for all g in G. So assume $g = 1$. First, if u lies in U, then we have $B = {}^{u}B$, and so B lies in ${}^{u}\Sigma_0$. Now let Σ be an any apartment of Δ containing B. There exists h in G such that $\Sigma = {}^{h}\Sigma_0$. We must have $B = {}^{hn}B$ for some n in N, because B is a chamber, and the set of chambers of ${}^{h}\Sigma_0$ is $\{ {}^{hn}B \mid n \in N \}$. Hence we have $hn \in \mathcal{N}(B) = B$, and so there exist u in U, t in T such that $hn = ut$, as $B = UT$. Therefore we get

$$\Sigma = {}^{h}\Sigma_0 = {}^{utn^{-1}}\Sigma_0 = {}^{u}\Sigma_0 ,$$

as tn^{-1} lies in N. Finally, if u, u' are elements of U such that ${}^{u}\Sigma_0 = {}^{u'}\Sigma_0$, then $u^{-1}u'$ normalizes Σ_0, and so $u^{-1}u'$ lies in $U \wedge N = \{1\}$. It follows that $u = u'$. ∎

Let n_0 be a fixed representative of w_0 in N. We have the following

<u>Proposition 3.1.2.</u> Let b be any element of B, and let P be a parabolic subgroup of G containing B. If ${}^{bn_0}P$ lies in ${}^{b}\Sigma_0$, then b lies in ${}^{n_0}P$.

<u>Proof</u> : Suppose that ${}^{bn_0}P$ lies in ${}^{b}\Sigma_0$. From the defining axioms of a Building, there exists an isomorphism $\gamma : \Sigma_0 \to {}^{b}\Sigma_0$ such that γ fixes all faces of B

and ${}^{a\circ}P$, as both B and ${}^{a\circ}P$ are in Σ_o and ${}^{b}\Sigma_o$. Let $\nu : \Sigma_o \longrightarrow {}^{b}\Sigma_o$ be the map defined by the position $X \mapsto {}^{b}X$ for every X in Σ_o . Then ν is an isomorphism , and it fixes all faces of B . We have $\gamma = \nu$ for Σ_o and ${}^{b}\Sigma_o$ are thin chamber complexes , and $X^{\gamma} = X = X^{\nu}$ for every face X of the chamber B of Σ_o . But then we get

$${}^{ba\circ}P = ({}^{a\circ}P)^{\nu} = ({}^{a\circ}P)^{\gamma} = {}^{a\circ}P \ .$$

So b lies in $\mathcal{N}({}^{a\circ}P) = {}^{a\circ}P$. ∎

The Weyl group W acts in a natural way on the set of roots Φ ([C_2] page 19) . The longest element w_o of W has the property of being the unique element of W such that $w_o(\Phi^+) = \Phi^-$. In particular , we have $w_o(\Pi) = -\Pi$. Also , for any i in $\{1,...,\ell\}$, there exists $j(i)$ in $\{1,...,\ell\}$ such that

$$w_o s_i w_o = s_{j(i)} \quad ([C_2] \text{ page } 55) \ .$$

Consequently $w_o(-\alpha_i) = \alpha_{j(i)}$, for every $i = 1,...,\ell$. The mapping $i \mapsto j(i)$ of I into itself is clearly involutory , in the sense that $j^2(i) = i$ for every i . It is called the opposition involution .

<u>Proposition 3.1.3.</u> Let s be a simple root and let r be the simple root $w_o(-s)$. For every g in G let R_g be the set of all apartments of Δ containing ${}^{g}B$ and ${}^{g n\circ}P_{(s)}$. Then the map given by

$$a \mapsto {}^{g x_r(a)}\Sigma_o$$

is a bijection of K onto R_g .

<u>Proof</u> : It is sufficient to consider the case when $g = 1$, for then the result follows for all g in G . By Proposition 3.1.1. , we only need to show that if u is an element of U such that ${}^{n\circ}P_{(s)}$ lies in ${}^{u}\Sigma_o$, then u lies in the root-subgroup X_r , as then we use the isomorphism $x_r : a \mapsto x_r(a)$ between $(K,+)$ and X_r . Now , if ${}^{n\circ}P_{(s)}$ lies in ${}^{u}\Sigma_o$, then u lies in ${}^{n\circ}P_{(s)}$, by Proposition 3.1.2. . Hence we get that u lies in $U \wedge {}^{n\circ}P_{(s)} = X_r$ ([C_2] page 59) , and we are done . ∎

Proposition 3.1.4. Let r' be a simple root, w an element of W and u an element of U. If we denote by r the root $w(r')$, then for every apartment Σ of Δ containing ${}^{u\dot{w}}B$ and ${}^{u\dot{w}n_0}P_{(w_0(-r'))}$, there exists a unique a in K such that $\Sigma = {}^{ux_r(a)}\Sigma_0$.

Proof : It is sufficient to consider the case when $u = 1$, for then the result follows for all u in U. From Proposition 3.1.3., there exists a' in K such that

$$\Sigma = {}^{\dot{w}x_{r'}(a')}\Sigma_0$$

Now we have $\dot{w}X_{r'}\dot{w}^{-1} = X_{w(r')} = X_r$ ([C_2] page 57), and so there exists a in K such that $\dot{w}x_{r'}(a')\dot{w}^{-1} = x_r(a)$. Hence we obtain

$$\Sigma = {}^{\dot{w}x_{r'}(a')}\Sigma_0 = {}^{\dot{w}x_{r'}(a')\dot{w}^{-1}}\Sigma_0 = {}^{x_r(a)}\Sigma_0 ,$$

as $\dot{w}$ lies in $\mathcal{N}(\Sigma_0)$. Uniqueness follows from the fact that $U \wedge N = \{1\}$. #

Observation 3.1.5. Suppose φ is an autoprojectivity of G. Then we know from Theorem 2.2.6., that φ acts on Δ. If now Σ is an apartment of Δ, we put

$$\Sigma^\varphi = \{ X^\varphi \mid X \in \Sigma \} .$$

It is then clear that Σ^φ is an apartment of Δ. In this way φ acts on the set of apartments of Δ. Suppose now that φ fixes the Borel subgroup B of G. It then follows that φ permutes the set of apartments of Δ containing B. Hence, by Proposition 3.1.1., we can define a map

$$\theta : U \longrightarrow U , \text{ by the rule } {}^{u^\theta}\Sigma_0 = ({}^u\Sigma_0)^\varphi .$$

We show that this map θ is a bijection. If we start with the inverse φ^{-1} of φ, then with the same procedure we can define a map

$$\zeta : U \longrightarrow U ; \text{ by the rule } {}^{u^\zeta}\Sigma_0 = ({}^u\Sigma_0)^{\varphi^{-1}} .$$

It turns out that $\theta\zeta = \zeta\theta = id_U$, and so θ is bijective.

Suppose now that φ fixes also ${}^{n_0}P_{\{s\}}$, for some simple root s. Then, by Proposition 3.1.3., we have $X_r^\theta \subseteq X_r$, where $r = w_0(-s)$. Again we note that φ^{-1} fixes ${}^{n_0}P_{\{s\}}$ as well, and so we also get $X_r^\zeta \subseteq X_r$, which implies $X_r^\theta =$

X_r . Therefore , for every simple root r , we have $X_r^{\varphi} = X_r$. ∎

In the previous chapter we proved that if φ is any autoprojectivity of G , then there exist an inner automorphism ι_g and a graph automorphism δ of G such that the autoprojectivity $\varphi(\iota_g^{-1}\delta^{-1})^*$ fixes T and every parabolic subgroup containing B , i.e. $\varphi(\iota_g^{-1}\delta^{-1})^*$ lies in $\Gamma_{[G/B],T}$. In the next paragraph we shall start studying the action of the group $\Gamma_{[G/B],T}$ on Δ .

§ 3.2 The action of $\Gamma_{[G/B],T}$ on the set of apartments containing B .

In this paragraph we shall prove the existence , for every φ in $\Gamma_{[G/B],T}$, of a diagonal automorphism d of G acting as φ does on the apartments Σ_0 and ${}^{x_r(1)}\Sigma_0$ for every simple root r .

For the rest of this chapter we shall denote by Γ_0 the group $\Gamma_{[G/B],T}$.

Proposition 3.2.1. Let φ be in Γ_0 . Then the automorphism of Δ induced by φ is type-preserving .

Proof : To show that the automorphism of Δ induced by φ is type-preserving , we have to show that for every X in Δ , X and X^{φ} are conjugate in G . So let X be in Δ , and let Σ be an apartment of Δ containing X and B (existence of Σ comes from the axioms of a building) . Σ^{φ} is an apartment containing B as well , and so , by Proposition 3.1.1. , there exist u , u' in U such that $\Sigma = {}^{u}\Sigma_0$, and $\Sigma^{\varphi} = {}^{u'}\Sigma_0$. We define two isomorphisms between Σ and Σ^{φ} . First we denote by ε the isomorphism induced by φ , and then we define ν by the map $Y \mapsto {}^{u'u^{-1}}Y$ for every Y in Σ .

Let now P be a parabolic subgroup containing B . Then we have

$$P^{\varepsilon} = P^{\varphi} = P ,$$

as φ lies in Γ_0 , and $P^{\nu} = P$ as $u'u^{-1} \in U \leq B$. Therefore we must have

$\varepsilon = \nu$. Hence we obtain $X^{\varphi} = X^{\varepsilon} = X^{\nu} = {}^{n'n^{-1}}X$, and so X and X^{φ} are conjugate . It follows that the automorphism of Δ induced by φ is type-preserving . #

Proposition 3.2.2. Let φ be in Γ_o , and let Σ be any apartment of Δ containing B . If φ fixes Σ , then it fixes every face of Σ . In particular φ fixes every face of Σ_o .

Proof : Suppose φ fixes Σ . The map $\varepsilon : \Sigma \longrightarrow \Sigma$ given by $X \mapsto X^{\varphi}$ for every X in Σ , defines an automorphism of the thin chamber complex Σ fixing all the faces of the chamber B . Then ε must be the identity on Σ . Therefore φ fixes all the elements of Σ . In particular , as $T^{\varphi} = T \Rightarrow$ $\Sigma_o{}^{\varphi} = \Sigma_o$, φ fixes all the faces of Σ_o . #

Let r be a simple root , and let s be the simple root $w_o(-r)$. If φ lies in Γ_o , then by Proposition 3.2.2. , we have ${}^{n_o}P_{(s)}{}^{\varphi} = {}^{n_o}P_{(s)}$, and so , by Observation 3.1.5. , we have $X_r{}^{\theta} = X_r$. Then there exists a unique k_r in K such that

$$x_r(1)^{\theta} = x_r(k_r) .$$

We must have k_r in K^{*} , as θ is bijective , and already $1^{\theta} = 1$. Let χ be the character of the root-lattice $\mathbb{Z}\Phi$ obtained by extending by linearity the map of Π into K^{*} given by the position $r \mapsto k_r$ for every r in Π . We have the following

Lemma 3.2.3. Let φ be in Γ_o . Then there exists a diagonal automorphism d of G such that

$$X^{\varphi} = X^{d}$$

for every face X of Δ lying in Σ_o or in ${}^{x_r(1)}\Sigma_o$, where r is any simple root .

Proof : Let d be the diagonal automorphism of G induced by the character χ of the root-lattice $\mathbb{Z}\Phi$, defined above . Then , by the definition of the automorphism d , we have

$$x_\alpha(k)^d = x_\alpha(\chi(\alpha)k) \quad \text{for every } \alpha \text{ in } \Phi \text{ and every } k \text{ in } K,$$

and

$$t^d = t \quad \text{for every } t \text{ in } T \ ([C_1] \text{ page } 100).$$

Hence we have $T^d = T$ and ${X_\alpha}^d = X_\alpha$ for every root α. In particular d fixes every parabolic subgroup P of G containing B, as P is generated by T and some of the root-subgroups X_α. It follows that the autoprojectivity d^* of G induced by d lies in Γ_o. Also we have $x_r(1)^d = x_r(\chi(r)) = x_r(k_r)$ for every r in Π, and so

$$({}^{x_r(1)}\Sigma_o)^d = {}^{x_r(1)^d}(\Sigma_o)^d = {}^{x_r(k_r)}\Sigma_o .$$

Let us denote by ψ the autoprojectivity $\varphi(d^*)^{-1}$. Then ψ lies in Γ_o, and so we have $X^\psi = X$ for every X in Σ_o, by Proposition 3.2.2. Hence we get

$$X^\varphi = X^{d^*} = X^d \quad \text{for every } X \text{ in } \Sigma_o .$$

Suppose now that r is a simple root. Then we have

$$({}^{x_r(1)}\Sigma_o)^\psi = ({}^{x_r(k_r)}\Sigma_o)^{(d^*)^{-1}} = ({}^{x_r(1)}\Sigma_o)^{d^*(d^*)^{-1}} = {}^{x_r(1)}\Sigma_o .$$

Therefore ψ fixes every face of ${}^{x_r(1)}\Sigma_o$ by Proposition 3.2.2.. So, for every r in Π and every X in ${}^{x_r(1)}\Sigma_o$, we have $X^\varphi = X^{d^*} = X^d$. #

We give now the following

Definition 3.2.4. : We define $\Gamma_{o,\Pi}$ to be the group of all autoprojectivities in Γ_o such that $({}^{x_r(1)}\Sigma_o)^\varphi = {}^{x_r(1)}\Sigma_o$ for every simple root r. #

Therefore, by Lemma 3.2.3., for any autoprojectivity φ in Γ_o, there exists a diagonal automorphism d of G such that the autoprojectivity $\varphi(d^*)^{-1}$ lies in $\Gamma_{o,\Pi}$.

In the next paragraph we shall take a closer look at the bijection θ.

§ 3.3 The bijection θ .

As we have already noticed , for every φ in Γ_o , the map $\theta : U \longrightarrow U$, given by the rule ${}^{u^\theta}\Sigma_o = ({}^u\Sigma_o)^\varphi$ is a bijection fixing the root-subgroups X_r for every simple root r . We also have that $1^\theta = 1$, as $\Sigma_o{}^\varphi = \Sigma_o$.

We shall now further investigate the properties of θ in order to define a particular field automorphism of G .

Proposition 3.3.1. Let φ be in Γ_o . Then we have

$$({}^uX)^\varphi = {}^{u^\theta}X$$

for every u in U and every X in Σ_o .

Proof : Let u be in U and X in Σ_o . By the definition of θ , we have ${}^{u^\theta}\Sigma_o = ({}^u\Sigma_o)^\varphi$, and so the map $A \mapsto A^\varphi$, where A lies in ${}^u\Sigma_o$, determines an isomorphism ε between ${}^u\Sigma_o$ and ${}^{u^\theta}\Sigma_o$. Let ν be the isomorphism of ${}^u\Sigma_o$ onto ${}^{u^\theta}\Sigma_o$ given by the map $A \mapsto {}^{u^\theta u^{-1}}A$, for every A in ${}^u\Sigma_o$. Then for every parabolic subgroup P containing B we have

$$P^\varepsilon = P^\varphi = P = P^\nu ,$$

and so we must have $\varepsilon = \nu$. Hence , if X is any face of Σ_o , we obtain

$$({}^uX)^\varphi = ({}^uX)^\nu = {}^{u^\theta u^{-1}u}X = {}^{u^\theta}X ,$$

and we are done . #

We are now able to prove a proposition and two corollaries which will be crucial later to define a field automorphism of G .

Proposition 3.3.2. Let φ be in Γ_o . Let r be a positive root , and u be an element of U . Then , for every a in K , there exists a unique b in K such that

$$(ux_r(a))^\theta = u^\theta x_r(b) .$$

Furthermore , we have $b = 0$ if and only if $a = 0$.

Proof : There exist a simple root r' and an element w in W such that r =

$w(r')$ ([C_1] Proposition 2.1.8.) . Let us denote by s the simple root $w_o(-r')$.
We apply Proposition 3.3.1. to the elements ${}^{\dot{w}}B$ and ${}^{\dot{w}n_o}P_{(s)}$ to get

$$({}^{u\dot{w}}B)^{\varphi} = {}^{u^{\theta}\dot{w}}B \quad \text{and} \quad ({}^{u\dot{w}n_o}P_{(s)})^{\varphi} = {}^{u^{\theta}\dot{w}n_o}P_{(s)} .$$

Hence φ induces a bijection of the set of all apartments containing ${}^{\dot{w}}B$ and ${}^{\dot{w}n_o}P_{(s)}$ onto the set of all apartments containing ${}^{u^{\theta}\dot{w}}B$ and ${}^{u^{\theta}\dot{w}n_o}P_{(s)}$.
Therefore, by Proposition 3.1.4. , for every a in K , there exists a unique b in K such that

$$({}^{ux_r(a)}\Sigma_o)^{\varphi} = {}^{u^{\theta}x_r(b)}\Sigma_o .$$

But , by the definition of θ , we have $({}^{ux_r(a)}\Sigma_o)^{\varphi} = {}^{(ux_r(a))^{\theta}}\Sigma_o$, and so we obtain

$$(ux_r(a))^{\theta} = u^{\theta}x_r(b) .$$

Suppose now b' is any element of K such that $(ux_r(a))^{\theta} = u^{\theta}x_r(b')$. Then we get $u^{\theta}x_r(b) = u^{\theta}x_r(b')$, which implies $x_r(b) = x_r(b')$, and so $b' = b$ as x_r is an isomorphism . Also , we have

$b = 0 \iff u^{\theta}x_r(b) = u^{\theta} \iff (ux_r(a))^{\theta} = u^{\theta} \iff ux_r(a) = u \iff x_r(a) = 1 \iff$
$a = 0$. #

<u>Corollary 3.3.3.</u> Let φ be in Γ_e , and let r be a positive root . Then , for every a in K , there exists a unique b in K such that

$$x_r(a)^{\theta} = x_r(b) .$$

Also , we have $b = 0$ if and only if $a = 0$.

<u>Proof</u> : This follows immediately from Proposition 3.3.2. by taking $u = 1$, as $1^{\theta} = 1$. #

<u>Corollary 3.3.4.</u> Let φ be in Γ_e , and u be an element of U . Let $r_1 , \ldots , r_m$ be pairwise distinct positive roots . Then , for every $(a_1 , \ldots , a_m)$ in K^m , there exists a unique $(b_1 , \ldots , b_m)$ in K^m such that

$$(ux_{r_1}(a_1) \cdots x_{r_m}(a_m))^{\theta} = u^{\theta}x_{r_1}(b_1) \ldots x_{r_m}(b_m) .$$

Furthermore , for any i in $\{1,\ldots,m\}$, we have $b_i = 0$ if and only if $a_i = 0$.

<u>Proof</u> : We shall prove the existence part by induction on m . The case in

which $m = 1$ comes from Proposition 3.3.2. . So assume $m > 1$, and suppose we proved the result for every k less than m . Now let $(a_1, ..., a_m)$ be in K^m . We put

$$u' = ux_{r_1}(a_1) \cdots x_{r_{m-1}}(a_{m-1}) .$$

Then , by Proposition 3.3.2. , there exists b_m in K such that $(u'x_{r_m}(a_m))^\theta = u'^\theta x_{r_m}(b_m)$, and , by induction assumption , there exists $(b_1, ..., b_{m-1})$ in K^{m-1} such that $u'^\theta = u^\theta x_{r_1}(b_1) \ldots x_{r_{m-1}}(b_{m-1})$. Hence we obtain $(ux_{r_1}(a_1) \cdots x_{r_m}(a_m))^\theta = u^\theta x_{r_1}(b_1) \ldots x_{r_m}(b_m)$. Uniqueness follows from the fact that every element of U is expressible in a unique way as a product $\prod_{r \in \Phi^+} x_r(c_r)$ for any fixed order on Φ^+ .

Suppose now that for some i in $\{1,...,m\}$ we have $a_i = 0$. Then we get $ux_{r_1}(a_1) \cdots x_{r_m}(a_m) = ux_{r_1}(a_1) \cdots x_{r_{i-1}}(a_{i-1})x_{r_{i+1}}(a_{i+1}) \cdots x_{r_m}(a_m)$, and so there exists $(c_1, ..., c_{m-1})$ be in K^{m-1} such that

$(ux_{r_1}(a_1) \cdots x_{r_m}(a_m))^\theta = u^\theta x_{r_1}(c_1) \ldots x_{r_{i-1}}(c_{i-1})x_{r_{i+1}}(c_i) \cdots x_{r_m}(c_{m-1})$.

But then , if we take $d_1 = c_1, \ldots , d_{i-1} = c_{i-1}, d_i = 0 , d_{i+1} = c_i , ..., d_m = c_{m-1}$, we get

$$(ux_{r_1}(a_1) \cdots x_{r_m}(a_m))^\theta = u^\theta x_{r_1}(d_1) \ldots x_{r_m}(d_m) .$$

Hence we must have $b_j = d_j$ for every $j = 1,...m$, for uniqueness . In particular we obtain $b_i = 0$. Similarly we can prove that $b_i = 0$ implies $a_i = 0$. #

We already knew that for every φ in Γ_0 we have $X_r^\theta = X_r$ for every simple root r . By Corollary 3.3.3. , we can now say that this equality holds for any positive root r . Hence , for every positive root r , we can define a bijection f_r of K onto itself by the map $x_r(f_r(a)) = (x_r(a))^\theta$. In the next paragraph we shall apply this to the case when φ is an element of $\Gamma_{0,\Pi}$. The the bijections f_r will have the following properties

$f_r(0) = 0$ for every positive root r ,

$f_r(1) = 1$ for every simple root r .

We shall prove that if the rank ℓ of G is at least 2, then the f_r's are all equal to an automorphism f of K.

§ 3.4 The maps f_r.

To investigate the properties of the maps f_r, we need to know the behaviour of the bijection θ with respect to decompositions of the group U in terms of root-subgroups. We shall use a lemma, that we prove in a more general context.

Let X be a group, and let $X_1, \ldots, X_n$ be subgroups of X such that each element x of X can uniquely be written as a product $x = x_1 \cdots x_n$, where x_i lies in X_i for every $i = 1, \ldots, n$. Suppose that we have

$[X_i, X] \leq X_{i+1} \cdots X_n$ for $i = 1, \ldots, n-1$, and $[X_n, X] = \{1\}$.

Then, in particular we have that $X_i \cdots X_n$ is a normal subgroup of X for every $i = 1, \ldots, n$. Hence, from [St] page 25, we get that for every σ in the symmetric group S_n, each element x of X can be uniquely expressed as a product $x = x_{\sigma_1} \ldots x_{\sigma_n}$, where each x_i lies in X_i. We now fix i in $\{1, \ldots, n\}$. Then, for every $j = 1, \ldots n$, there exists a unique y_j in X_j, such that

$x = y_{\sigma_i} y_{\sigma_1} \cdots y_{\sigma_{i-1}} y_{\sigma_{i+1}} \cdots y_{\sigma_n}$,

as this is precisely the decomposition of x corresponding to the permutation $\nu = \sigma(i\ i-1 \ldots 3\ 2\ 1)$ (i.e. $\nu_1 = \sigma_i, \nu_2 = \sigma_1, \ldots$). We shall show that $y_{\sigma_i} = x_{\sigma_i}$.

<u>Lemma 3.4.1.</u> Let X, σ and i be defined as above. Let x be an element of X. We consider the decomposition $x = x_{\sigma_1} \ldots x_{\sigma_n}$, where each x_j lies in X_j. If the decomposition of x relative to the permutation $\nu = \sigma(i\ i-1 \ldots 3\ 2\ 1)$ (i.e. $\nu_1 = \sigma_i, \nu_2 = \sigma_1, \ldots$) is $x = y_{\nu_1} y_{\nu_2} \ldots y_{\nu_n}$, where each y_j lies in X_j, then we have $y_{\nu_1} = x_{\sigma_i}$. Hence we have

$x = x_{\sigma_1} \cdots x_{\sigma_n} = x_{\sigma_i} y_{\sigma_1} \cdots y_{\sigma_{i-1}} y_{\sigma_{i+1}} \cdots y_{\sigma_n}$.

Proof : We shall proceed by induction on n . If $n = 1$, then there is nothing to prove . So let n be greater than 1 , and assume the result for $n-1$. Let j in $\{1,\dots,n\}$ be such that $\sigma_j = n$. Suppose first that $i = j$. Then we have $n = \sigma_i$, and so x_{σ_i} lies in $X_n \leq Z(X)$. Hence we have

$$x = x_{\sigma_1} \cdots x_{\sigma_n} = x_{\sigma_i} x_{\sigma_1} \cdots x_{\sigma_{i-1}} x_{\sigma_{i+1}} \cdots x_{\sigma_n} ,$$

and we are done . So let us assume $i \neq j$. Hence σ_i is less than n . Let us define τ in S_n by the rule

$$\tau_k = \begin{cases} \sigma_k & \text{if } 1 \leq k < j \\ \sigma_{k+1} & \text{if } j \leq k < n \\ n & \text{if } k = n . \end{cases}$$

To apply the induction assumption , we consider the group $Z = X/X_n$, and its subgroups $Z_k = X_k X_n/X_n$ ($\cong X_k$) for every $k = 1,\dots,n-1$. Then we have $Z = Z_1 \cdots Z_{n-1}$ and each element z of Z can uniquely be written as a product $z = z_1 \cdots z_{n-1}$, where z_k lies in Z_k for every $k = 1,\dots,n-1$. Besides , if $k = 1,\dots,n-2$ we have

$$[Z_k , Z] = [X_k , X] X_n/X_n \leq X_{k+1} \cdots X_n /X_n = Z_{k+1} \cdots Z_{n-1} ,$$

and $[Z_{n-1} , Z] = \{1\}$.

For every u in X let us denote by $\bar{u}$ the corresponding element uX_n of Z . By the definition of τ , we have

$$x = x_{\sigma_1} \cdots x_{\sigma_n} = x_{\tau_1} \cdots x_{\tau_n} ,$$

and so we get

$$\bar{x} = \bar{x}_{\tau_1} \cdots \bar{x}_{\tau_{n-1}} ,$$

as x_{τ_n} lies in X_n . Now we have $\tau_n = n$, and so we can consider the restriction $\bar{\tau}$ of τ to the set $\{1,\dots,n-1\}$. We fix k in $\{1,\dots,n-1\}$, and we consider the decomposition $v_{\vartheta_1} \cdots v_{\vartheta_{n-1}}$ of $\bar{x}$ with respect to the permutation $\vartheta = \bar{\tau}(k\ k-1 \dots 3\ 2\ 1)$ (i.e. $\vartheta_1 = \bar{\tau}_k$, $\vartheta_2 = \bar{\tau}_1$, ...) , where v_j lies in Z_j , for every $j = 1,\dots,n-1$. Then we have

$$v_{\vartheta_1} = \bar{x}_{\tau_k} ,$$

by the induction assumption . Now , for every $j = 1,\dots,n-1$, there exists a unique u_j in X_j such that $v_j = u_j X_n$. Therefore we get

$$x = u_{\varphi_1} \cdots u_{\varphi_{n-1}} \mod X_n ,$$

and so there exists a unique u_n in X_n such that $x = u_{\varphi_1} \cdots u_{\varphi_{n-1}} u_n$. We note that the condition $v_{\varphi_1} = \bar{x}_{\tau_k}$, implies $u_{\varphi_1} = x_{\tau_k}$, as both u_{φ_1} and x_{τ_k} lie in X_{τ_k} for, by definition of $\bar{v}$, we have $\varphi_1 = \tau_k$. Finally, if we take

$$k = i \text{ if } i < j \text{ , and } k = i-1 \text{ if } i > j ,$$

then we get

$$x = x_{\sigma_i} u_{\sigma_1} \cdots u_{\sigma_{i-1}} u_{\sigma_{i+1}} \cdots u_{\sigma_n} ,$$

as $u_{\sigma_j} = u_{\tau_n} = u_n$ lies in $X_n \leq Z(X)$. Hence we obtain the decomposition

$$x = x_{v_1} u_{v_2} \cdots u_{v_n} ,$$

with respect to the permutation ν. But, by hypothesis, the decomposition of x with respect to ν is $x = y_{v_1} \cdots y_{v_n}$, and so we must have

$$y_{v_1} = x_{v_1} = x_{\sigma_i}$$

for uniqueness of decomposition. #

We apply the previous lemma to the case when X is the unipotent radical U of B.

We fix a total order $<$ on Φ^+ compatible with the height function , and we number the elements in Φ^+ such that $r_1 < r_2 < \dots < r_N$. Then we have

$$U = X_{r_1} \cdots X_{r_N} ,$$

with uniqueness of expression,

$$[X_{r_i}, U] \leq X_{r_{i+1}} \cdots X_{r_N} \text{ for every } i = 1,\dots,N-1 \text{, and } [X_{r_N}, U] = \{1\} .$$

We can therefore apply the lemma. For short in the following we shall write X_j for X_{r_j}, for every $j = 1,\dots,N$.

Let σ be in S_N, and let u be any element of U. Then we can express u uniquely as a product $u = x_{\sigma_1} \cdots x_{\sigma_N}$, where x_j lies in X_j for every $j = 1,\dots,N$. We shall show that for any φ in Γ_e, we have $u^\theta = x_{\sigma_1}{}^\theta \cdots x_{\sigma_N}{}^\theta$, where θ is the bijection of U induced by φ.

So let φ be an autoprojectivity in Γ_e and let i be fixed in $\{1,\dots,N\}$. By Lemma 3.4.1., we get

$$u = x_{\sigma_i} y_{\sigma_1} \cdots y_{\sigma_{i-1}} y_{\sigma_{i+1}} \cdots y_{\sigma_N} .$$

where for every j in $\{1,\ldots,N\}$, $j \neq i$, we have y_{σ_j} in X_{σ_j}. Hence, by Corollary 3.3.4., for every j in $\{1,\ldots,N\}$, $j \neq i$ there exists u_{σ_j} in X_{σ_j} such that

$$u^{\theta} = x_{\sigma_i}{}^{\theta} u_{\sigma_1} \cdots u_{\sigma_{i-1}} u_{\sigma_{i+1}} \cdots u_{\sigma_N} .$$

Now u^{θ}, as an element of U, has a unique decomposition

$$u^{\theta} = v_{\sigma_1} \cdots v_{\sigma_N} ,$$

with v_j in X_j for every $j = 1,\ldots,N$. Again we apply the lemma to obtain

$$u^{\theta} = v_{\sigma_i} t_{\sigma_1} \cdots t_{\sigma_{i-1}} t_{\sigma_{i+1}} \cdots t_{\sigma_N}$$

with t_{σ_j} in X_{σ_j} for every j in $\{1,\ldots,N\}$, $j \neq i$. By Corollary 3.3.3., $x_{\sigma_i}{}^{\theta}$ lies in X_{σ_i}, and so

$$u^{\theta} = x_{\sigma_i}{}^{\theta} u_{\sigma_1} \cdots u_{\sigma_{i-1}} u_{\sigma_{i+1}} \cdots u_{\sigma_N} \quad \text{and} \quad u^{\theta} = v_{\sigma_i} t_{\sigma_1} \cdots t_{\sigma_{i-1}} t_{\sigma_{i+1}} \cdots t_{\sigma_N}$$

are two ways of decomposing u^{θ} with respect to the same total order of Φ^+. Hence these two decompositions must coincide, for uniqueness of decomposition. In particular we obtain

$$x_{\sigma_i}{}^{\theta} = v_{\sigma_i} .$$

But i was any fixed element of $\{1,\ldots, N\}$, and so we get

$$x_{\sigma_i}{}^{\theta} = v_{\sigma_i} \quad \text{for every } i = 1,\ldots,N .$$

Therefore we get

$$u^{\theta} = v_{\sigma_1} \cdots v_{\sigma_N} = x_{\sigma_1}{}^{\theta} \cdots x_{\sigma_N}{}^{\theta} .$$

We can then prove the following

<u>Proposition 3.4.2.</u> Let $<$ be any total order on Φ^+, and let us number the elements in Φ^+ such that $r_1 < r_2 < \ldots < r_N$. Let φ be in Γ_o, and θ be the bijection of U induced by φ. If

$$u = x_1 \cdots x_N$$

is the decomposition of the element u of U with x_i in X_{r_i} for every $i = 1,\ldots,N$, then the decomposition of u^{θ} with respect to the same order of Φ^+ is

$$u^{\theta} = x_1{}^{\theta} \ldots x_N{}^{\theta}$$

(with $x_i{}^{\theta}$ in X_{r_i} for every $i = 1,\ldots,N$).

<u>Proof</u>: Let $<'$ be any total order on Φ^+ compatible with the height function.

Then there exists σ in S_N such that we have

$$r_{\sigma_1} <' r_{\sigma_2} <' \ldots <' r_{\sigma_N} .$$

Let us denote by s_i the root r_{σ_i} for every $i = 1,\ldots,N$, so that we get $s_1 <' s_2 <' \ldots <' s_N$. Let ν be the inverse of σ. Then, by the previous discussion, if $u = y_{\nu_1} \cdots y_{\nu_N}$ is the decomposition of u with y_i in X_{s_i} for every $i = 1,\ldots,N$, we have

$$u^\theta = y_{\nu_1}{}^\theta \cdots y_{\nu_N}{}^\theta .$$

Now, for any $i = 1,\ldots,N$, y_i in X_{s_i} implies y_{ν_i} in $X_{s_{\nu_i}}$, which leaves us with y_{ν_j} in X_{r_j}, for every $j = 1,\ldots,N$. So $u = x_1 \cdots x_N$ and $u = y_{\nu_1} \cdots y_{\nu_N}$ are two decompositions of u with x_j and y_{ν_j} in X_{r_j}, for every $j = 1,\ldots,N$. But then we must have

$$y_{\nu_j} = x_j \quad \text{for every } j = 1,\ldots,N ,$$

for uniqueness of decomposition. Therefore we obtain

$$u^\theta = y_{\nu_1}{}^\theta \cdots y_{\nu_N}{}^\theta = x_1{}^\theta \cdots x_N{}^\theta ,$$

and we are done. ∎

We can now start proving that the maps f_r for r in Φ^+, previously defined, are in fact all the same.

<u>Proposition 3.4.3.</u> Let r, s be simple roots such that $N_{r,s} = -1$. Then we have

$$f_r = f_s = f_{r+s} ,$$

and this map is multiplicative.

<u>Proof</u> : From the fact that $N_{r,s} = -1$, it follows that $r+s$ is a root, and so f_{r+s} is actually well defined. Let now a, b be elements of K. By the commutator formula, we have

$$x_s(b)x_r(a) = x_r(a)x_s(b)x_{r+s}(ab) \cdots$$

Thus, by Proposition 3.4.2., we have

$$x_s(b)^\theta x_r(a)^\theta = x_r(a)^\theta x_s(b)^\theta x_{r+s}(ab)^\theta \cdots$$

and so we get

$$x_s(f_s(b))x_r(f_r(a)) = x_r(f_r(a))x_s(f_s(b))x_{r+s}(f_{r+s}(ab)) \cdots$$

But, applying again the commutator formula, we also have

$$x_s(f_s(b))x_r(f_r(a)) = x_r(f_r(a))x_s(f_s(b))x_{r+s}(f_r(a)f_s(b)) \cdots$$

Therefore we obtain

$$f_{r+s}(ab) = f_r(a)f_s(b)$$

by uniqueness of expression. As the maps f_r are induced by φ which lies in $\Gamma_{0,\Pi}$, we have $f_r(1) = 1 = f_s(1)$, and so

$$f_r(a) = f_r(a) \cdot 1 = f_r(a)f_s(1) = f_{r+s}(a),$$

and similarly for f_s. Hence we obtain

$$f_r = f_{r+s} = f_s.$$

Besides we have

$$f_r(ab) = f_{r+s}(ab) = f_r(a)f_s(b) = f_r(a)f_r(b),$$

and so f_r is multiplicative. #

Proposition 3.4.4. Let r, s be simple roots such that $N_{r,s} = -1$. Then the map $f = f_r$ ($= f_s$) is additive.

Proof: By Proposition 3.4.3., we have $f = f_r = f_s = f_{r+s}$. Applying several times the commutator formula, we get

$$x_s(a)x_r(1)x_{r+s}(b) = x_r(1)x_s(a)x_{r+s}(a) \cdots x_{r+s}(b) = \ldots =$$
$$= x_r(1)x_s(a)x_{r+s}(a+b) \cdots$$

Hence, by Proposition 3.4.2. and 3.4.3., we have

$$x_s(f(a))x_r(1)x_{r+s}(f(b)) = x_r(1)x_s(f(a))x_{r+s}(f(a+b)) \cdots$$

But, applying again the commutator formula, we also have

$$x_s(f(a))x_r(1)x_{r+s}(f(b)) = x_r(1)x_s(f(a))x_{r+s}(f(a)+f(b)) \cdots,$$

and so we must have

$$f(a+b) = f(a) + f(b)$$

by uniqueness of expression. #

We finally have the following

Proposition 3.4.5. Let r, s be simple roots such that $N_{r,s} = -1$. Then the map $f = f_r$ ($= f_s$) is a field automorphism of K.

Proof : This comes from Proposition 3.4.3. and 3.4.4., as we already know that the maps f_r are bijective (for the map $\theta : X_r \longrightarrow X_r$ is a bijection from Observation 3.1.5.). ∎

From now on we shall assume that the rank ℓ of the simple algebraic group G is at least 2.

Our aim is to show that the maps f_r are all the same for every positive root r, and that this map is a field automorphism of G. Hence we need only to show that for every r, s in Φ^+ we have $f_r = f_s$, for then this map will automatically be a field automorphism, by Proposition 3.4.5..

Proposition 3.4.6. Let r, s be positive roots. Then we have

$f_r = f_s$.

Proof : As G is a simple algebraic group, its Dynkin diagram is a connected graph. Then, if α, β are simple roots, there exists a sequence of simple roots $\alpha_1,\dots,\alpha_m$, such that $\alpha = \alpha_1$, $\beta = \alpha_m$, and α_i, α_{i+1} are not orthogonal for every $i = 1,\dots,m-1$. Hence we have $N_{\alpha_i,\alpha_{i+1}} = \pm 1$. Now, in general, we have $N_{\delta,\gamma} = -N_{\gamma,\delta}$ for any δ, γ in Φ ([C$_1$] Theorem 4.1.2.), and so we can apply Proposition 3.4.3. to get

$f_{\alpha_i} = f_{\alpha_{i+1}}$ for every $i = 1,\dots,m-1$.

Therefore we obtain $f_\alpha = f_\beta$. But α, β were arbitrary simple roots, and so we can define a function f (which we already know is an automorphism of K) such that

$f = f_\alpha$ for every simple root α.

We shall show that $f = f_r$ for every positive root r, by induction on the height ht(r) of r. If ht(r) = 1, then r is a simple root, and we are done. So now suppose that ht(r) is greater than 1, and assume the result for every positive root s of height less than ht(r). By [C$_1$] Lemma 3.6.2., there exists a simple root α such that $r-\alpha$ is still a positive root. Let $r_0 = r - m\alpha$ be the first term of the α-chain through r. Then we have $m \geq 1$, by the choice of α. Also r_0 is a positive root because if we consider the decomposition $r = \sum n_\gamma \gamma$ (sum over the simple roots), then we have

$n_\gamma \geq 0$ for every simple root γ, and there must exist a simple root $\beta \neq \alpha$ such that $n_\beta \neq 0$, as $ht(r) > 1$, and r is not a simple root. Now

$r = r_0 + m\alpha$ and $m \geq 1$ imply

$$ht(r_0) < ht(r),$$

and so we get

$$f_{r_0} = f$$

by the induction assumption. Also, by the commutator formula, we get

$$x_\alpha(1)x_{r_0}(a) = x_{r_0}(a)x_\alpha(1) \cdots x_r(\eta a) \cdots,$$

where $\eta = (-1)^m M_{r_0,\alpha,m}$. But we have $M_{r_0,\alpha,m} = \pm 1$, as $r_0 - \alpha$ does not lie in Φ (by the choice of m). Hence we obtain $\eta = \pm 1$. Applying the bijection θ, we get

$$x_\alpha(1)x_{r_0}(f(a)) = x_{r_0}(f(a))x_\alpha(1) \cdots x_r(f_r(\eta a)) \cdots .$$

But, from the commutator formula, we have

$$x_\alpha(1)x_{r_0}(f(a)) = x_{r_0}(f(a))x_\alpha(1) \cdots x_r(\eta f(a)) \cdots ,$$

and therefore we are left with

$$f_r(\eta a) = \eta f(a),$$

for uniqueness of expression. Hence we obtain

$$f_r(a) = f(a),$$

because f is an automorphism of K (here we need only that f is an automorphism of the additive group of K). Therefore we have $f_r = f$. #

Thus, for any autoprojectivity φ in $\Gamma_{0,\Pi}$, we got an automorphism f of the field K such that

$$x_r(a)^\theta = x_r(f(a)) \text{ for every } r \text{ in } \Phi^+ \text{ and every } a \text{ in } K,$$

where θ is the bijection of U induced by φ.

Let us denote by F the field automorphism of G induced by f. We recall that F is defined by the map

$x_r(a)^F = x_r(f(a))$ for every r in Φ and every a in K.

We shall refer to the automorphism F simply as to the field automorphism of G induced by φ, where φ lies in $\Gamma_{0,\Pi}$.

Proposition 3.4.7. Let φ be in $\Gamma_{0,\Pi}$, and let F be the field automorphism of G induced by φ. Then we have

$$({}^u\Sigma_0)^\varphi = ({}^u\Sigma_0)^F$$

for any u in U.

Proof : Let u be in U. Let $r_1 < ... < r_N$ be any fixed total order on Φ^+. Then there exists $k_1, ..., k_N$ in K such that

$$u = x_{r_1}(k_1) ... x_{r_N}(k_N).$$

By Proposition 3.4.2. we have $u^\theta = x_{r_1}(k_1)^\theta ... x_{r_N}(k_N)^\theta$, and so

$$u^\theta = x_{r_1}(f(k_1)) ... x_{r_N}(f(k_N))$$

by Proposition 3.4.6. and the definition of f. But, by the definition of F, we have

$$u^F = x_{r_1}(f(k_1)) ... x_{r_N}(f(k_N))$$

as well, and so we obtain $u^\theta = u^F$. If now we prove that $\Sigma_0{}^F = \Sigma_0$, then we shall have

$$({}^u\Sigma_0)^\varphi = {}^{u^\theta}\Sigma_0 = {}^{u^F}\Sigma_0 = {}^{u^F}(\Sigma_0)^F = ({}^u\Sigma_0)^F,$$

and we shall be done. Now, by the definition of F, we get

$$U^F = U \text{ and } (U^-)^F = U^-,$$

and so

$$B^F = B \text{ and } (B^-)^F = B^-,$$

as $B = \mathcal{N}(U)$ and $B^- = \mathcal{N}(U^-)$. Hence we get

$$T^F = (B \wedge B^-)^F = B \wedge B^- = T.$$

Therefore we have $\Sigma_0{}^F = \Sigma_0$. #

We finally are able to prove the following

Lemma 3.4.8. Let φ be in $\Gamma_{0\Pi}$, and let the rank of G be at least 2. Then there exists a field automorphism F of G such that we have $X^{\varphi} = X^{F}$ for every face X of Δ.

<u>Proof</u> : Let F be the field automorphism of G induced by φ. We first observe that the autoprojectivity F^* of G induced by F lies in $\Gamma_{0,\Pi}$. We already know that F^* lies in $\Gamma_{B,T}$, as we proved that F fixes both B and T. From the fact that $X_r^{F} = X_r$ for every root r, it follows in particular that F fixes every parabolic subgroup of G containing B, and so F^* lies in $\Gamma_{[G/B],T}$ $= \Gamma_0$. Finally, for every simple root r, we have

$$({}^{x_r(1)}\Sigma_0)^{F^*} = {}^{x_r(1)^F}\Sigma_0 = {}^{x_r(1)}\Sigma_0 ,$$

as $x_r(1)^F = x_r(f(1)) = x_r(1)$. Hence F^* lies in $\Gamma_{0,\Pi}$.

Now let X be any face of Δ. Then, by the defining axioms of a Building, there exists an apartment Σ of Δ such that B and X are both in Σ. By Proposition 3.1.1., there exists u in U such that $\Sigma = {}^u\Sigma_0$. Let us denote by ψ the autoprojectivity $\varphi(F^{-1})^*$ of G. Then ψ lies in $\Gamma_{0,\Pi}$, and we have

$$({}^u\Sigma_0)^{\psi} = {}^u\Sigma_0$$

by Proposition 3.4.7.. Therefore, by Proposition 3.2.2., we get

$$Y^{\psi} = Y \text{ for every face } Y \text{ of } {}^u\Sigma_0 .$$

In particular we obtain $X^{\psi} = X$, as X lies in ${}^u\Sigma_0$. Hence we have

$$X^{\varphi} = X^{F^*} = X^{F} .$$ #

We are now able to prove the following

<u>Theorem 3.4.9.</u> Let G be a simple algebraic group over the field $K = \bar{F}_p$, where p is any prime. Then, if the rank of G is at least 2, for every autoprojectivity φ of G there exist an inner automorphism ι_g, a graph automorphism δ, a diagonal automorphism d and a field automorphism F of G such that the autoprojectivity

$$\varphi\,((Fd\delta\iota_g)^{-1})^*$$

fixes all the faces of the building Δ canonically associated to G.

<u>Proof</u> : The result now comes immediately, by applying Lemma 2.3.2., 2.4.9.,

3.2.3. and 3.4.8. . #

Now that we know that for any autoprojectivity φ of G there exists an automorphism α of G such that φ and α act in the same way on the building Δ, we prove the uniqueness of α.

Proposition 3.4.10. Let G be a simple algebraic group over K. If α is an automorphism of G fixing every face of Δ, then α is the identity.

Proof : First, as α fixes B and B^-, it must fix also U and U^-. Now let u be in U. Then we have

$$ {}^u\Sigma_0 = ({}^u\Sigma_0)^\alpha = {}^{u^\alpha}\Sigma_0 , $$

which gives $u^\alpha = u$ by uniqueness. Similarly one can prove that for every apartment Σ of Δ containing B^-, there exists a unique u in U^-, such that $\Sigma = {}^u\Sigma_0$. Again then we get $u^\alpha = u$, for every u in U^-. It follows that

$$ g^\alpha = g \text{ for every } g \text{ in } G, $$

as $G = \langle U, U^- \rangle$. Hence α is the identity. #

Remark : The previous result holds in the more general case when G is semisimple, as the crucial point is that $G = \langle U, U^- \rangle$ which holds in fact in the case when G is semisimple.

Corollary 3.4.11. In the hypothesis of Theorem 3.4.9., for any autoprojectivity φ of G, there exists a unique automorphism α of G such that φ and α act in the same way on the building Δ.

Proof : Existence follows from Theorem 3.4.9., by taking $\alpha = Fd\delta\iota_g$. Uniqueness then follows from Proposition 3.4.10. . #

We are also able to obtain the well known structure of the automorphism group of G.

Corollary 3.4.12. In the hypothesis of Theorem 3.4.9., for every automorphism α of G, there exists an inner automorphism ι_g, a graph automorphism δ, a diagonal automorphism d and a field automorphism F of G such that $\alpha = Fd\delta\iota_g$.

Proof : By Theorem 3.4.9., there exists an inner automorphism ι_g, a graph automorphism δ, a diagonal automorphism d and a field utomorphism F of G such that α and $Fd\delta\iota_g$ act in the same way on the Building Δ. Then we must have $\alpha = Fd\delta\iota_g$ by Proposition 3.4.10.. #

As a corollary of Proposition 3.4.10., we also obtain

Corollary 3.4.13. Let G be a simple algebraic group over K. Then the homomorphism

$* : \mathrm{Aut}\, G \longrightarrow \mathrm{Aut}\, \mathcal{L}(G)$

which assigns to every automorphism α of G the autoprojectivity α^* of G induced by α, is injective.

Proof : Let α be an automorphism of G fixing every subgroup of G. Then, in particular, α fixes every face of Δ, and so it is the identity by Proposition 3.4.10. (This result follows also directly from a result by Cooper ([Co] 2.2.2.) that says that the homomorphism $*$ is injective for every perfect abstract group G, but in our case we can give a proof from the previous discussion). #

Remark : From the previous remark, it is then clear that Corollary 3.4.13. holds also in the case when G is semisimple (It doesn't always work for reductive groups, as one can see by taking G to be a torus, and α to be the inversion automorphism).

We shall therefore identify the groups $\mathrm{Aut}\, G$ and $(\mathrm{Aut}\, G)^*$. We also give the following

Definition 3.4.14. For every simple algebraic group G we define Γ to be the kernel of the action of the group of autoprojectivities of G on the Building Δ, i.e.

$$\Gamma = \{ \varphi \in \mathrm{Aut}\, \mathcal{L}(G) \mid X^{\varphi} = X \quad \forall X \in \Delta \} .$$

We shall call the elements of Γ *exceptional autoprojectivities* of G. ∎

Corollary 3.4.11. then implies

Corollary 3.4.15. Let G be a simple algebraic group over the field K. Then, if the rank of G is at least 2, we have

$$\mathrm{Aut}\, \mathcal{L}(G) = \Gamma \rtimes \mathrm{Aut}\, G .$$ ∎

We shall see in Chapter 5, using different arguments, that this relation holds also when G has rank 1.

Remark : The procedure we followed gives actually a constructive way to obtain the automorphism α acting on Δ as the autoprojectivity φ does, for, a posteriori, one has that if φ lies in $\Gamma_{0,\Pi}$, then the field automorphism F induced by φ acts on U in the same way as the bijection θ induced by φ does. Similarly it is possible to define a bijection θ^- on U^- and show that F coincides with θ^- on U^-. Hence F is completely determined by θ and θ^-. It is also clear that the bijection θ is an automorphism of U.

Chapter 4 The case rank G at least 3 .

In the previous chapter we showed that for any simple algebraic group G over K of rank at least 2 , the group Aut $L(G)$ is the semidirect product of the group Γ and the group Aut G . Hence the problem whether every autoprojectivity of G is induced by an automorphism of G is reduced to study in which cases the group Γ coincides with the identity subgroup of Aut $L(G)$.

In this chapter we shall show that this happens if the rank of G is at least 3 , and the characteristic p of K is not 2 .

§ 4.1 Some properties of the group Γ .

To prove Corollary 3.4.15. , we fixed a Borel subgroup B of G and a maximal torus T of G contained in B . We defined subgroups $\Gamma_{B,T}$, $\Gamma_{[G/B],T} = \Gamma_0$, $\Gamma_{0\Pi}$ and finally Γ depending on the choice of the couple (B,T) . But clearly Γ does not depend on the choice of (B,T) . It will be useful , for later use , to summarize some properties of the exceptional autoprojectivities of G (i.e. the autoprojectivities of G lying in Γ) , properties which can be directly deduced from results we got in the previous chapter .

Proposition 4.1.1. Let φ be in Γ . Then φ fixes every maximal torus and every maximal unipotent subgroup of G . Also , if $\{ X_\alpha \mid \alpha \in \Phi \}$ is the set of root-subgroups of G relative to the choice of a maximal torus T , then we have

$$X_\alpha^{\varphi} = X_\alpha$$

for every α in Φ .

Proof : Let T be a maximal torus of G . Let B be a Borel subgroup of G containing T , and let B^- be its opposite relative to T . Then we have $T = B \wedge B^-$, and so

$$T^{\varphi} = (B \wedge B^-)^{\varphi} = B \wedge B^- = T .$$

Now let U be a maximal unipotent subgroup of G. Then U is the set of unipotent elements of a Borel subgroup B of G (in fact $B = \mathcal{N}(U)$). Hence we get $U^\varphi = U$ as $B^\varphi = B$. Finally let T be a maximal torus of G, and let $\{ X_\alpha \mid \alpha \in \Phi \}$ be the set of root-subgroups of G relative to the choice of T. Let α be in Φ. Then there exists a Borel subgroup B of G containing T such that α lies in the fundamental system Π of Φ relative to the choice of B. Then we have $X_\alpha = U \wedge P^{n_0}$, where P is the parabolic subgroup $\langle B, n_\beta \rangle$, n_β is any representative in $\mathcal{N}(T)$ of the fundamental reflection s_β and β is given by the relation $w_0 s_\alpha w_0 = s_\beta$, where w_0 is the longest element of the Weyl group $\mathcal{N}(T)/T$ ([C_2] page 59). So we get

$$X_\alpha^\varphi = (U \wedge P^{n_0})^\varphi = U \wedge P^{n_0} = X_\alpha . \qquad \#$$

Observation 4.1.2. We note that in general, to show that an autoprojectivity φ of a group H is the identity, we only need to show that $\langle x \rangle^\varphi = \langle x \rangle$ for every x in H. If H is a torsion group it will be even enough to restrict our attention to prime-power elements x of H.
In our case it will be therefore enough to show that we have

$\langle s \rangle^\varphi = \langle s \rangle$ for every semisimple element s of G and

$\langle u \rangle^\varphi = \langle u \rangle$ for every unipotent element u of G. #

We shall show that if the rank of G is at least 3, then every φ in Γ fixes every subgroup of G generated by a semisimple element. To do this we first consider in general autoprojectivities of maximal tori of G.

§ 4.2 Autoprojectivities of tori.

In this paragraph we consider in general autoprojectivities of tori. Incidentally we shall also fill the gap left in Theorem 2.1.6., where we considered only non-commutative reductive groups, by showing that the autoprojectivities of a one dimensional torus are almost all not index-preserving .

So let T be a torus , of dimension ℓ say . Then T is isomorphic to the direct product of ℓ copies of the multiplicative group K^* of K . Let q be any prime . For every torsion group X we denote by X_q the q-component of X. In our case we shall have $K^*_q \cong C_{q^\infty}$ for every $q \neq p$, and $K^*_p = \{1\}$. Therefore we get

$T_q \cong C_{q^\infty} \times \dots \times C_{q^\infty}$ (ℓ copies) if $q \neq p$, and

$T_p = \{1\}$.

Proposition 4.2.1. Let T be a torus of dimension ℓ . If ℓ is at least 2 , then every autoprojectivity of T is index-preserving . If ℓ is 1 , then every non-trivial autoprojectivity of T is not index-preserving .

Proof : Suppose first $\ell \geq 2$. Let φ be an autoprojectivity of T . To show that φ is index-preserving , it is enough to show that for every x in T of prime order q , we have $|\langle x\rangle^\varphi| = q$. So let x in T have prime order q . As $\ell \geq 2$, there exists y in T such that $\langle x, y\rangle \cong C_q \times C_q$. Therefore $\langle x,y\rangle^\varphi$ is a P-group ([S] Proposition 1.4 page 11) , and it is abelian as it is contained in T . Hence $\langle x,y\rangle^\varphi$ is an elementary abelian r-group for some prime r , and we must have $r = q$, as both $\langle x,y\rangle$ and $\langle x,y\rangle^\varphi$ have $q+1$ minimal subgroups . In particular we get $|\langle x\rangle^\varphi| = q$, and φ is index-preserving .
Suppose now $\ell = 1$. Then we have $T \cong K^*$, and so $T \cong \mathrm{Dr}\, C_{q^\infty}$ (product over all primes $q \neq p$) . Hence we obtain $L(T) \cong \mathrm{Cr}\, L(C_{q^\infty})$ ([S] Theorem 4 page 5) . We observe that for every q the lattice $L(C_{q^\infty})$ is a chain ($\{1\} < C_q < C_{q^2} < \dots$) , and so C_{q^∞} has only the trivial autoprojectivity . Therefore , for every permutation σ of the set of all primes different from p , we have a unique autoprojectivity φ_σ of $\mathrm{Dr}\, C_{q^\infty}$ (product over all primes $q \neq p$) such that $C_{q^\infty}{}^{\varphi_\sigma} = C_{\sigma(q)^\infty}$, and these are all the autoprojectivities of $\mathrm{Dr}\, C_{q^\infty}$ (thus $\mathrm{Aut}\, L(\mathrm{Dr}\, C_{q^\infty}) \cong S_{\aleph}$) . In particular every non-trivial autoprojectivity of $\mathrm{Dr}\, C_{q^\infty}$ is not index-preserving , and so the same holds for T . #

By the previous proposition we can say that Theorem 2.1.6. holds also in the case when G is a connected abelian reductive algebraic group of rank at least 2, while it drastically fails if the rank is 1.

If the dimension of the torus T is at least 3 we can say much more.

Proposition 4.2.2. Let T be a torus of dimension ℓ. If ℓ is at least 3, then every autoprojectivity of T is induced by an automorphism of T.

Proof : Let φ be an autoprojectivity of T. By Proposition 4.2.1., we have

$T_q^{\varphi} = T_q$ for every prime q.

Let us denote by φ_q, for every prime q, the restriction of φ to $L(T_q)$. Then φ_q is an autoprojectivity of the abelian q-group T_q. We have

$T_q \cong C_{q^\infty} \times \dots \times C_{q^\infty}$ (ℓ copies) for every prime $q \neq p$.

Hence, as $\ell \geq 3$, for every prime $q \neq p$ there exists, by a theorem by Baer ([S] Theorem 2 page 35), an automorphism f_q of T_q inducing φ_q. Then, as

$T = \oplus T_q$ (sum over all primes q different from p),

if we define $f = \oplus f_q$ (sum over all primes q different from p) in the obvious way, we get that f is an automorphism of T inducing φ. (Note that f is not unique, as for instance the inverse automorphism induces the identity autoprojectivity). #

We recall that if X is any group, then an automorphism α of X is called a power-automorphism of X if, for every x in X, there exists n in $\mathbb{Z}$ such that $x^{\alpha} = x^{n}$. We denote by $\mathcal{P}(X)$ the group of power-automorphisms of X. It is then clear that $\mathcal{P}(X)$ is the kernel of the homomorphism

$* : \mathrm{Aut}\, X \longrightarrow \mathrm{Aut}\, L(X)$ which assigns to every automorphism α of X the autoprojectivity α^* of X induced by α.

Suppose now that φ lies in Γ, and let T be any maximal torus of G. If the rank of G is at least 3, then, by Proposition 4.1.1., 4.2.2., there exists an automorphism f of T inducing φ on $L(T)$. We shall prove that f is in fact a power-automorphism of T . To do this , we shall consider separately simple algebraic groups G of different isogeny types .

We conclude this paragraph with the following

<u>Lemma 4.2.3.</u> Let F be an automorphism of the group $K^* \times K^*$. Let us denote by D the diagonal $\{ (a,a) \mid a \in K^* \}$ of $K^* \times K^*$. If F fixes the subgroups

$K^* \times \{1\}$, $\{1\} \times K^*$ and D of $K^* \times K^*$,

then , for every α in $\mathbb{N}$, there exists n in $\{ 1,\dots,p^{\alpha}-1 \}$ such that

$$s^F = s^n$$

for every s in $\mathbb{F}_{p^\alpha}^* \times \mathbb{F}_{p^\alpha}^*$.

<u>Proof</u> : Let α be in $\mathbb{N}$. Let k be an element of K^* such that $\langle k \rangle = \mathbb{F}_{p^\alpha}^*$. Then there exist a unique n_1 and a unique n_2 in $\{ 1,\dots,p^{\alpha}-1 \}$ such that

$$(k,1)^F = (k^{n_1},1) \text{ and } (1,k)^F = (1,k^{n_2}) ,$$

as F fixes $K^* \times \{1\}$ and $\{1\} \times K^*$. But F fixes also D , and so , as $(k,k)^F = (k^{n_1},k^{n_2})$, we must have $k^{n_1} = k^{n_2}$. Hence we get $n_1 = n_2$ by the choice of n_1 , n_2 . Call this common value n . Now let s be in $\mathbb{F}_{p^\alpha}^* \times \mathbb{F}_{p^\alpha}^*$. Then there exist r , s in $\mathbb{Z}$ such that $s = (k^r,k^s)$. We therefore have

$$s^F = (k^r,1)^F(1,k^s)^F = (k^{rn},1)\ (1,k^{sn}) = (k^{rn},k^{sn}) = s^n ,$$

and we are done . #

§ 4.3 The adjoint case .

In this paragraph we shall assume that G is a simple algebraic group over K of adjoint type (Lemma 4.3.7. excluded). We shall prove that if the rank of G is at least 3 , then every autoprojectivity in Γ fixes every subgroup of G generated by a semisimple element .

We introduce some notation that we shall use also later on . Let T be any maximal torus of G. We denote by X the character group of T. Therefore X is the group of all algebraic homomorphisms from T into K^*. X is a free abelian group of rank equal the dimension of T. Let Φ be the set of roots relative to T. Let B be a Borel subgroup of G containing T and let $\Pi = \{\alpha_1,\dots,\alpha_\ell\}$ be the fundamental system of Φ relative to the choice of B. As we suppose G to be adjoint, we have $X = \mathbb{Z}\Phi$.

We consider the map

$$T \times X \longrightarrow K^*$$

given by $(t,\chi) \mapsto \chi(t)$ for every t in T and every χ in X.

For every closed subgroup S of T we define $S^\perp$ by

$$S^\perp = \{\chi \in X \mid \chi(s) = 1 \ \ \forall s \in S\},$$

and for every subgroup A of X we define $A^\perp$ by

$$A^\perp = \{t \in T \mid \alpha(t) = 1 \ \ \forall \alpha \in A\}.$$

It turns out that $A^\perp$ is a closed subgroup of T ([C_2] page 26). We are interested in the family of subgroups $\{S^\perp \mid S$ is a subtorus of $T\}$ of X.

Let us denote by $\mathcal{A}$ the set of all subgroups A of X such that X/A is torsion free (i.e. X/A is free), and by $\mathcal{S}$ the set of all subtori of T. We have the following well known result:

<u>Proposition 4.3.1.</u> The map given by the position $A \mapsto A^\perp$ for every A in $\mathcal{A}$, is a bijection of $\mathcal{A}$ onto $\mathcal{S}$, its inverse being the map $S \mapsto S^\perp$ for every S in $\mathcal{S}$. Also we have

$$\dim A^\perp = \operatorname{rank}(X/A)$$

for every A in $\mathcal{A}$.

<u>Proof</u>: This follows from the fact that for every torus S we have

$$\dim S = \operatorname{rank} \operatorname{Hom}(S,K^*),$$

and from the fact that a diagonalizable group S is a torus if and only if $\operatorname{Hom}(S,K^*)$ is a free abelian group ([Sp] Proposition 2.5.8). ∎

Definition 4.3.2. Let I be the set $\{1,\dots,\ell\}$. For every $J \subseteq I$, we denote by A_J the subgroup $<\alpha_i \mid i \in J>$ of X. ∎

Proposition 4.3.3. For every subset J of I the group $A_J^{\perp}$ is a subtorus of T of dimension $\ell - |J|$.

Proof : We have $A_J = \bigoplus_{i \in J} \mathbb{Z}\alpha_i$, and so we get

$$X/A_J = \bigoplus_{i \in I} \mathbb{Z}\alpha_i / \bigoplus_{i \in J} \mathbb{Z}\alpha_i \cong \bigoplus_{i \in J'} \mathbb{Z}\alpha_i ,$$

where $J' = I \setminus J$. Therefore A_J lies in $\mathcal{A}$, and so $A_J^{\perp}$ is a subtorus of T of dimension $\ell - |J|$ by Proposition 4.3.1.. ∎

We shall now introduce some one-dimensional subtori of T.

Definition 4.3.4. For every i in I we define

$T_{\alpha_i} = \{ t \in T \mid \alpha_j(t) = 1 \ \forall j \in I, j \neq i \}$. ∎

Proposition 4.3.5. T_{α_i} is a one-dimensional subtorus of T for every i in I.

Proof : Let i be in I. We observe that $T_{\alpha_i} = A_J^{\perp}$, where $J = I \setminus \{i\}$. Then the result follows from Proposition 4.3.3.. ∎

Proposition 4.3.6. We have $T = T_{\alpha_1} \times \dots \times T_{\alpha_\ell}$.

Proof : Let us consider the direct product $T_{\alpha_1} \times \dots \times T_{\alpha_\ell}$. This is an ℓ-dimensional torus. If now we consider the map

$\pi : T_{\alpha_1} \times \dots \times T_{\alpha_\ell} \longrightarrow T$

given by $(t_1,\dots,t_\ell) \mapsto t_1 \cdots t_\ell$ where t_i lies in T_{α_i} for every i in I, then we get that π is an algebraic homomorphism. Let $(t_1,\dots,t_\ell)$ be in $\ker \pi$. By the definition of the subtori T_{α_i}, we have

$\alpha_r(T_{\alpha_s}) = \{1\}$ for every r, s in I such that $r \neq s$.

Let us fix i in I. We get

$$1 = \alpha_i(1) = \alpha_i(t_1 \cdots t_\ell) = \alpha_i(t_i).$$

Therefore we have $\alpha_j(t_i) = 1$ for every j in I and so we must have $t_i = 1$, as G is adjoint. Hence the algebraic homomorphism π is injective. But then it must be also surjective, because $\dim T = \ell = \dim T_{\alpha_1} \times \ldots \times T_{\alpha_\ell}$. Therefore we have $T = T_{\alpha_1} \times \ldots \times T_{\alpha_\ell}$. #

Note : As far as now, we can only say that the decomposition $T = T_{\alpha_1} \times \ldots \times T_{\alpha_\ell}$ coming from the above proposition, is just of abstract groups, because we did not prove that π^{-1} is an algebraic map. In fact this is so, and this will come from the following result, that we prove in a more general context.

For the moment let us suppose that G is a semisimple algebraic group over an algebraically closed field K. Let T be a maximal torus of G, and $(\chi_1,\ldots,\chi_\ell)$, $(\gamma_1,\ldots,\gamma_\ell)$ be dual $\mathbb{Z}$-bases resp. of the character group X and of the cocharacter group Y of T, in the usual duality $X \times Y \longrightarrow \mathbb{Z}$. Let us denote by T_i the subgroup $\{ t \in T \mid \chi_j(t) = 1 \text{ for every } j \neq i \}$, and by L_i the subgroup $\gamma_i(K^*)$ of T. Then both T_i and L_i are 1-dimensional subtori of T. For every $i = 1,\ldots,\ell$, we consider the algebraic homomorphisms

$\mu_i : T_i \longrightarrow K^*$ given by $\mu_i(t) = \chi_i(t)$ for every t in T_i, and

$\lambda_i : K^* \longrightarrow L_i$ given by $\lambda_i(k) = \gamma_i(k)$ for every k in K^*.

Then we have

Lemma 4.3.7. With the above notation for every $i = 1,\ldots,\ell$, we have $T_i = L_i$, and μ_i, λ_i are one the inverse of the other (in particular they are algebraic isomorphisms).

Proof : Let i be in $\{1,\ldots,\ell\}$. We recall that for every χ in X and every γ in Y, we have

$$k^{\langle\chi,\gamma\rangle} = \chi\gamma(k) \quad \text{for every } k \text{ in } K^*.$$

Suppose now that x is an element of L_i. Then there exists k in K^* such that

$x = \gamma_i(k)$. Thus, for every $j \neq i$, we get

$\chi_j(x) = \chi_j\gamma_i(k) = k^{<\chi_j, \gamma_i>} = k^0 = 1$,

so that x lies in T_i. Hence we have $L_i \leq T_i$ and so $L_i = T_i$ as they are both connected. Also we have

$\chi_i\gamma_i(k) = k^{<\chi_i, \gamma_i>} = k$ for every k in K,

so that $\chi_i\gamma_i = id_{K^*}$. We observe that χ_i is injective, as if t is in $\ker \chi_i$, we have $\chi_j(t) = 1$ for every j in $\{1,\dots,\ell\}$, so that $t = 1$. Therefore μ_i is bijective, and so we must have $\mu_i^{-1} = \lambda_i$, which also implies that λ_i is bijective with inverse μ_i. #

We now go back to the case when G is simple adjoint over $K = \overline{F}_p$. Let us denote by ξ_i the restriction to T_{α_i} of the algebraic homomorphism $\alpha_i : T \longrightarrow K^*$. We have the following

<u>Proposition 4.3.8.</u> For every i in I the map ξ_i is an algebraic isomorphism from T_{α_i} onto K^*.

<u>Proof</u> : As G is adjoint, $(\alpha_1,\dots,\alpha_\ell)$ is a $\mathbb{Z}$-basis of X. Therefore we can find in Y a dual basis $(\gamma_1,\dots,\gamma_\ell)$ of $(\alpha_1,\dots,\alpha_\ell)$. Then the result comes from Lemma 4.3.7.. #

We shall denote by ν_i the inverse of ξ_i for every i in I.

<u>Definition 4.3.9.</u> For every i, j in I such that $\alpha_i+\alpha_j$ lies in Φ, we define

$$S_{ij} = \Big(\bigwedge_{\substack{k\in I\\ k\neq i,j}} \ker \alpha_k\Big) \quad \text{and} \quad T_{ij} = S_{ij} \wedge \ker(\alpha_i+\alpha_j).$$

<u>Proposition 4.3.10.</u> Let i, j be elements of I such that $\alpha_i+\alpha_j$ lies in Φ. Then T_{ij} is a 1-dimensional subtorus of T.

<u>Proof</u> : Let us denote by A_{ij} the subgroup $<\alpha_k, \alpha_i+\alpha_j \mid k \in I, k \neq i, j>$.

Then we have

$$A_{ij} = (\bigoplus_{\substack{k \in I \\ k \neq i,j}} \mathbb{Z}\alpha_k) \oplus \mathbb{Z}(\alpha_i+\alpha_j) .$$

Hence we get

$$X/A_{ij} \cong (\mathbb{Z}\alpha_i \oplus \mathbb{Z}\alpha_j)/\, \mathbb{Z}(\alpha_i+\alpha_j) \cong \mathbb{Z} .$$

From the definition of T_{ij} , we have $T_{ij} = A_{ij}^{\perp}$, and so T_{ij} is a 1-dimensional subtorus by Proposition 4.3.1. . #

We shall now show that all the subtori of T previously defined are fixed by every element of Γ . For this purpose we shall use the following

Proposition 4.3.11. Let φ be in Γ . Then we have

$(\ker \alpha)^{\varphi} = \ker \alpha$

for every root α .

Proof : Let α be a root . We note that for every non-trivial u in the root-subgroup X_α , we have $\ker \alpha = T \wedge C(u)$. For let u be any non-trivial element of X_α . There exists a unique k in $K\setminus\{0\}$ such that $u = x_\alpha(k)$. Then , for t in T , we have

$u = tut^{-1} \iff x_\alpha(k) = x_\alpha(\alpha(t)k) \iff \alpha(t) = 1 \iff t \in \ker \alpha$.

Hence we have $\ker \alpha = T \wedge C(u)$.

Fix any non-trivial element u in X_α . By Proposition 4.1.1. , we have $X_\alpha^{\varphi} = X_\alpha$, and so there exists a non-trivial $\bar{u}$ in X_α such that $< u >^{\varphi} = < \bar{u} >$. Let now s be in $\ker \alpha$. As $T^{\varphi} = T$, there exists $\bar{s}$ in T such that $< s >^{\varphi} = < \bar{s} >$. The group $< u , s >$ is cyclic , as $(|u|, |s|) = 1$. Therefore $< \bar{u} , \bar{s} > = < u , s >^{\varphi}$ is cyclic as well , and so $\bar{s}$ lies in $C(\bar{u})$. Hence $\bar{s}$ lies in $T \wedge C(\bar{u}) = \ker \alpha$. We have so obtained $(\ker \alpha)^{\varphi} \leq \ker \alpha$. Similarly we can prove $(\ker \alpha)^{\varphi^{-1}} \leq \ker \alpha$, and so we get $(\ker \alpha)^{\varphi} = \ker \alpha$. #

Proposition 4.3.12. All the subtori T_{α_i} , T_{ij} previously defined , are fixed by every element of Γ .

Proof : It is enough to obverve that all these subtori are elements of the sublattice of $L(T)$ spanned by the set $\{ \ker \alpha \mid \alpha \in \Phi \}$, and then use Proposition 4.3.11. . ∎

We introduce , for every i , j in I such that $\alpha_i + \alpha_j$ lies in Φ , the isomorphism

$$\rho_{ij} : K^* \times K^* \to T_{\alpha_i} \times T_{\alpha_j}$$

given by the map $(a,b) \mapsto \nu_i(a)\nu_j(b^{-1})$ for every a ,b in K^* . ρ_{ij} is an algebraic isomorphism by Proposition 4.3.8. .

We have the following

Proposition 4.3.13. Let us denote by D the diagonal $\{ (k,k) \mid k \in K^* \}$ of $K^* \times K^*$. Then , for every i , j in I such that $\alpha_i+\alpha_j$ lies in Φ , we have

$$D^{\rho_{ij}} = T_{ij} .$$

Proof : It is enough to show that $D^{\rho_{ij}} \leq T_{ij}$, as ρ_{ij} is injective , D and T_{ij} are 1-dimensional connected algebraic groups , and ρ_{ij} is an algebraic map . So let a be an element of K^* . We have to show that the element $t = \nu_i(a)\nu_j(a^{-1})$ of T lies in T_{ij} . Let m be in I , $m \neq i$, j . Then we have

$$\alpha_m(\nu_i(a)) = \alpha_m(\nu_j(a^{-1})) = 1$$

as $\alpha_m(T_{\alpha_r}) = \{1\}$ for every $r \neq m$. Therefore t lies in S_{ij} . Also we have

$$(\alpha_i+\alpha_j)(\nu_i(a)\nu_j(a^{-1})) = \alpha_i\nu_i(a)\,\alpha_j\nu_j(a^{-1})$$

again because $\alpha_s(T_{\alpha_r}) = \{1\}$ for every $s \neq r$. But , for every s in I and for every k in K^* , we have $\alpha_s\nu_s(a) = \xi_s\nu_s(a) = a$, by the definition of ξ_s . Hence we get

$$(\alpha_i+\alpha_j)(t) = \alpha_i\nu_i(a)\,\alpha_j\nu_j(a^{-1}) = aa^{-1} = 1 .$$

So t lies in $T_{ij} = S_{ij} \wedge \ker(\alpha_i+\alpha_j)$. ∎

Let now φ be an element of Γ . Then , by Proposition 4.1.1. , for any maximal torus T of G , we have $T^{\varphi} = T$ and so , by Proposition 4.2.2. , as the dimension of T is at least 3 , there exists an automorphism f of T inducing φ on $L(T)$. We are now able to prove the following

Proposition 4.3.14. Let φ be in Γ, and let T be any maximal torus of G. Then any automorphism of T inducing φ on $L(T)$ is a power-automorphism.

Proof : Let f be an automorphism of T inducing φ on $L(T)$. Let t be in T. From Proposition 4.3.6., we have $T = T_{\alpha_1} \times \dots \times T_{\alpha_\ell}$, and so, for every i in I, there exists a unique t_i in T_{α_i} such that $t = t_1 \cdots t_\ell$. Let us denote by k_i the element $\xi_i(t_i)$ of $K^\times$ for every i in I, and let F_{p^α} be the subfield of K generated by $k_1, \dots, k_\ell$. Let also k be a generator of the multiplicative group of F_{p^α}. From Proposition 4.3.12., we have $T_{\alpha_i}{}^{\varphi} = T_{\alpha_i}$ for every i in I, and so we get

$$T_{\alpha_i}{}^{f} = T_{\alpha_i} \quad \text{for every } i \text{ in } I.$$

Hence, for every i in I, there exists a unique n_i in $\{1, \dots, p^\alpha - 1\}$ such that $\nu_i(k)^f = \nu_i(k^{n_i})$. Let now i, j be in I such that $\alpha_i + \alpha_j$ lies in Φ. By Proposition 4.3.12., we have

$$T_{ij}{}^{f} = T_{ij}.$$

We consider the automorphism $f_{ij} : T_{\alpha_i} \times T_{\alpha_j} \longrightarrow T_{\alpha_i} \times T_{\alpha_j}$ induced by f. Let us denote by F_{ij} the composite

$$\rho_{ij} f_{ij} \rho_{ij}^{-1} : K^\times \times K^\times \longrightarrow K^\times \times K^\times ,$$

where the isomorphisms ρ_{ij} are those defined above. Then F_{ij} is an automorphism of $K^\times \times K^\times$. We have $(K^\times \times \{1\})^{\rho_{ij}} = T_{\alpha_i}$, $(\{1\} \times K^\times)^{\rho_{ij}} = T_{\alpha_j}$ by the definition of ρ_{ij} and $D^{\rho_{ij}} = T_{ij}$ by Proposition 4.3.13.. From the fact that $T_{\alpha_i}{}^{f} = T_{\alpha_i}$, $T_{\alpha_j}{}^{f} = T_{\alpha_j}$ and $T_{ij}{}^{f} = T_{ij}$, it follows that

$$(K^\times \times \{1\})^{F_{ij}} = K^\times \times \{1\}, \quad (\{1\} \times K^\times)^{F_{ij}} = \{1\} \times K^\times \quad \text{and} \quad D^{F_{ij}} = D.$$

Hence we can apply Lemma 4.2.3. to obtain n in $\mathbb{N}$ such that

$$(a,b)^{F_{ij}} = (a,b)^n \quad \text{for every } a, b \text{ in } F_{p^\alpha}{}^\times.$$

We therefore get $(k,1)^n = (k,1)^{F_{ij}} = (k^{n_i},1)$ and $(1,k)^n = (1,k)^{F_{ij}} = (1,k^{n_j})$. It follows that

$$n_i \equiv n \equiv n_j \mod p^\alpha - 1,$$

and so $n_i = n_j$ by the choice of n_i and n_j. Hence we have $n_i = n_j$ for every i, j in I such that $\alpha_i + \alpha_j$ lies in Φ. As the Dynkin diagram of G is connected, we conclude that $n_i = n_j$ for every i, j in I. Call this common value n, so that we have

$\nu_i(k)^f = \nu_i(k^n)$ for every i in I.

Now we go back to our element t in T. We had $t = t_1 \cdots t_\ell$, with t_i in T_{α_i} for every i in I, and we denoted by k_i the element $\xi_i(t_i)$. There exists $r_1,\dots,r_\ell$ in $\mathbb{Z}$ such that $k_i = k^{r_i}$ for every i in I. Hence we obtain

$$t = t_1 \cdots t_\ell = \nu_1(k_1) \cdots \nu_\ell(k_\ell) = \nu_1(k^{r_1}) \dots \nu_\ell(k^{r_\ell}),$$

and so

$$t^f = \nu_1(k^{r_1})^f \dots \nu_\ell(k^{r_\ell})^f = \nu_1(k^{nr_1}) \dots \nu_\ell(k^{nr_\ell}) = t_1^n \cdots t_\ell^n = t^n.$$

Therefore f is a power-automorphism of T. #

Corollary 4.3.15. Let s be a semisimple element of the simple algebraic group of adjoint type G of rank at least 3. Then we have

$$\langle s \rangle^\varphi = \langle s \rangle$$

for every φ in Γ.

Proof : Let φ be in Γ. There exists a maximal torus T of G such that s lies in T. By Proposition 4.2.2., 4.3.14., there exists a power-automorphism f of T inducing φ on $\mathcal{L}(T)$. Therefore we have $\langle s \rangle^\varphi = \langle s \rangle^f = \langle s \rangle$. #

In the next paragraph we shall prove that the same result holds also when G is simply-connected.

§ 4.4 The simply-connected case.

In this paragraph we shall always assume that G is a simply-connected simple algebraic group of rank at least 3.

First we introduce some notation.

Let T be any maximal torus of G. We denote by Y the cocharacter group of T, i.e. the group $\mathrm{Hom}(K^*, T)$ of all algebraic homomorphisms of K^* into T. Let Φ be the set of roots of G relative to T, let B be a Borel subgroup of

G containing T and let $\Pi = \{\alpha_1,\ldots,\alpha_\ell\}$ be the fundamental system of Φ relative to the choice of B. For every α in Φ there is a unique $\alpha^\vee$ in Y such that $<\alpha,\alpha^\vee> = 2$ in the duality $X \times Y \to \mathbb{Z}$ ([Sp] 8.1.2., 9.1.5., [C_2] page 19). $\alpha^\vee$ is called the coroot corresponding to α. If we denote by $\Phi^\vee$ the set of coroots, then the map $\alpha \mapsto \alpha^\vee$ defines a bijection between Φ and $\Phi^\vee$. It turns out that the set $\Pi^\vee = \{\alpha_1^\vee,\ldots,\alpha_\ell^\vee\}$ is a fundamental system of $\Phi^\vee$ ([C_1] page 50). In our case, as G is simply-connected, we have $Y = \mathbb{Z}\Phi^\vee$, and so $\Pi^\vee$ is a $\mathbb{Z}$-basis of Y.

Proposition 4.4.1. Let $(\gamma_1,\ldots,\gamma_\ell)$ be any $\mathbb{Z}$-basis of Y. Then we have

$T = \gamma_1(K^*) \times \cdots \times \gamma_\ell(K^*)$.

Proof : This follows from the fact that there exists an isomorphism $Y \otimes K^* \to T$ sending $\gamma \otimes k$ to $\gamma(k)$ for every γ in Y and every k in K^* ([C_2] Proposition 3.1.2.). #

Definition 4.4.2. For every α in Φ, we define $L_{\alpha^\vee}$ to be the subgroup $\alpha^\vee(K^*)$ of T. #

$L_{\alpha^\vee}$ is therefore a 1-dimensional subtorus of T for every α in Φ.

Corollary 4.4.3. We have $T = L_{\alpha_1^\vee} \times \cdots \times L_{\alpha_\ell^\vee}$.

Proof : This follows from Proposition 4.4.1., as $\{\alpha_1^\vee,\ldots,\alpha_\ell^\vee\}$ is a $\mathbb{Z}$-basis of Y. #

As we noticed for the decomposition $T = T_{\alpha_1} \times \ldots \times T_{\alpha_\ell}$ in the adjoint case, for the moment we can only say that the decomposition $T = L_{\alpha_1^\vee} \times \cdots \times L_{\alpha_\ell^\vee}$ we got in the previous proposition is of abstract groups. In fact this is a decomposition of T as an algebraic group, as this incidentally will come from the following argument.

We need an explicit bijective algebraic homomorphism from K^* into $L_{\alpha_i^\vee}$ for every i in I.

Definition 4.4.4. For every i in I, we denote by ζ_i the map $K^* \longrightarrow L_{\alpha_i^\vee}$ given by $\zeta_i(k) = \alpha_i^\vee(k)$ for every k in K^*. ∎

It is then clear that ζ_i is an algebraic epimorphism for every i in I. But, in view of Lemma 4.3.7., we can say even more.

Proposition 4.4.5. ζ_i is an algebraic isomorphism for every i in I.

Proof : As G is simply-connected, $\{\alpha_1^\vee,\ldots,\alpha_\ell^\vee\}$ is a $\mathbb{Z}$-basis of Y, and so we can find in X a dual basis $\{\chi_1,\ldots,\chi_\ell\}$ of $\{\alpha_1^\vee,\ldots,\alpha_\ell^\vee\}$. Then the result comes from Lemma 4.3.7.. ∎

For every i, j in I such that $\alpha_i+\alpha_j$ lies in Φ, we define the map

$$\mu_{ij} : K^* \times K^* \longrightarrow L_{\alpha_i^\vee} \times L_{\alpha_j^\vee}$$

by $(a,b) \mapsto \zeta_i(a)\zeta_j(b)$ for every a, b in K^*.

Then, by Proposition 4.4.5., μ_{ij} is an algebraic isomorphism for every i, j in I such that $\alpha_i+\alpha_j$ lies in Φ.

Proposition 4.4.6. If we denote by D the diagonal $\{ (k,k) \mid k \in K^*\}$ of $K^* \times K^*$, then, for every i, j in I such that $\alpha_i+\alpha_j$ lies in Φ, we have

$$D^{\mu_{ij}} = L_{\alpha_i^\vee+\alpha_j^\vee} .$$

Proof : First we note that $\alpha_i+\alpha_j$ lies in Φ if and only if $\alpha_i^\vee+\alpha_j^\vee$ lies in $\Phi^\vee$, as we have

$$< \alpha_i,\alpha_j^\vee > = 0 \iff < \alpha_j,\alpha_i^\vee > = 0 \quad ([C_2] \text{ page } 23) .$$

Therefore $L_{\alpha_i^\vee+\alpha_j^\vee}$ is well defined. But then we have

$$L_{\alpha_i^\vee+\alpha_j^\vee} = (\alpha_i^\vee+\alpha_j^\vee)(K^*) ,$$

which coincides with $D^{\mu_{ij}}$ by the definition of μ_{ij} . #

We shall now show that all the subtori $L_{\alpha^\vee}$ of T previously defined are fixed by every element of Γ . For this purpose we shall use the following

Proposition 4.4.7. Let α be in Φ . Then we have

$$L_{\alpha^\vee} = T \wedge \langle X_\alpha , X_{-\alpha} \rangle .$$

Proof : $L_{\alpha^\vee}$ is a 1-dimensional torus , and so it is a maximal torus of $\langle X_\alpha , X_{-\alpha} \rangle$, which is a simple algebraic group of rank 1 . Let n_α be any representative of the reflection w_α of $W = \mathcal{N}(T)/T$ in $\langle X_\alpha , X_{-\alpha} \rangle$ ([C$_2$] page 19) . Then we have the Bruhat decomposition

$$\langle X_\alpha , X_{-\alpha} \rangle = L_{\alpha^\vee} X_\alpha \cup L_{\alpha^\vee} X_\alpha n_\alpha X_\alpha$$

of $\langle X_\alpha , X_{-\alpha} \rangle$. But $L_{\alpha^\vee} X_\alpha n_\alpha X_\alpha \cap T = \emptyset$, as $B \cap Bn_\alpha B = \emptyset$, and so we have

$$T \wedge \langle X_\alpha , X_{-\alpha} \rangle \subseteq T \cap (L_{\alpha^\vee} X_\alpha) = L_{\alpha^\vee} .$$

Hence we get

$$L_{\alpha^\vee} = T \wedge \langle X_\alpha , X_{-\alpha} \rangle ,$$

as clearly $L_{\alpha^\vee} \subseteq T \wedge \langle X_\alpha , X_{-\alpha} \rangle$. #

Proposition 4.4.8. Let α be in Φ . Then , for every φ in Γ , we have

$$L_{\alpha^\vee}{}^{\varphi} = L_{\alpha^\vee} .$$

Proof : Let φ be in Γ . By Proposition 4.4.7. we have $L_{\alpha^\vee} = T \wedge \langle X_\alpha , X_{-\alpha} \rangle$, and so we get

$$L_{\alpha^\vee}{}^{\varphi} = T^{\varphi} \wedge \langle X_\alpha{}^{\varphi}, X_{-\alpha}{}^{\varphi} \rangle = T \wedge \langle X_\alpha , X_{-\alpha} \rangle = L_{\alpha^\vee} ,$$

by Proposition 4.1.1. . #

Let now φ be an element of Γ . Then , by Proposition 4.1.1. , for any maximal torus T of G , we have $T^{\varphi} = T$ and so , by Proposition 4.2.2. , as the dimension of T is at least 3 , there exists an automorphism f of T

inducing φ on $L(T)$. We are now able to prove the following

<u>Proposition 4.4.9.</u> Let φ be in Γ, and let T be any maximal torus of G. Then any automorphism of T inducing φ on $L(T)$ is a power-automorphism.

<u>Proof</u> : The proof is similar to the proof of Proposition 4.3.14. . The only difference is that here, for every i, j in I such that $\alpha_i+\alpha_j$ lies in Φ, we use the automorphism V_{ij} of $K^* \times K^*$ given by the composite $\mu_{ij} f_{ij} \mu_{ij}^{-1}$ (where the isomorphism μ_{ij} is the one defined above, and $f_{ij} : L_{\alpha_i^\vee} \times L_{\alpha_j^\vee} \longrightarrow L_{\alpha_i^\vee} \times L_{\alpha_j^\vee}$ is the automorphism induced by f). We have

$$(K^*\times\{1\})^{\mu_{ij}} = L_{\alpha_i^\vee} \text{ and } (\{1\}\times K^*)^{\mu_{ij}} = L_{\alpha_j^\vee}$$

by the definition of μ_{ij} and

$$D^{\mu_{ij}} = L_{\alpha_i^\vee+\alpha_j^\vee},$$

by Proposition 4.4.6. . From the fact that $L_{\alpha_i^\vee}{}^f = L_{\alpha_i^\vee}$, $L_{\alpha_j^\vee}{}^f = L_{\alpha_j^\vee}$ and $L_{\alpha_i^\vee+\alpha_j^\vee}{}^f = L_{\alpha_i^\vee+\alpha_j^\vee}$, it follows that

$$(K^*\times\{1\})^{V_{ij}} = K^*\times\{1\},\ (\{1\}\times K^*)^{V_{ij}} = \{1\}\times K^* \text{ and } D^{V_{ij}} = D.$$

Then we can proceed as in the proof of Proposition 4.3.14. . #

<u>Corollary 4.4.10.</u> Let s be a semisimple element of the simple simply-connected algebraic group G of rank at least 3. Then we have

$$< s >^\varphi = < s >$$

for every φ in Γ.

<u>Proof</u> : Let φ be in Γ. There exists a maximal torus T of G such that s lies in T. By Proposition 4.2.2. , 4.4.9. , there exists a power-automorphism f of T inducing φ on $L(T)$. Therefore we have $< s >^\varphi = < s >^f = < s >$. #

In the next paragraph we shall deal with the cases left out so far.

§ 4.5 The case when G is neither adjoint nor simply-connected .

In this paragraph we finally consider the case when G is neither adjoint nor simply-connected . Such a group is therefore forced to have rank at least 3 , and to have type A_ℓ or D_ℓ . In particular , the Dynkin diagram of G has only single bonds .

Let T be a maximal torus of G . Let X , Φ , Π , Y , $\Phi^\vee$ and $\Pi^\vee$ be defined as in the previous paragraph . For every subgroup A of Y we denote by $A^\wedge$ the subtorus $<\gamma(K^*) \mid \gamma \in A>$ of T . $A^\wedge$ is a subtorus as it is closed and connected , being the union of closed and connected subgroups of T . Note that we have

$$A^\wedge = a_1(K^*) \cdots a_r(K^*)$$

whenever $\{a_1,\ldots,a_r\}$ is a set of generators of A . We have the following

<u>Proposition 4.5.1.</u> Let A be a subgroup of Y of rank r . Then the dimension of $A^\wedge$ is r . In particular we have

$$T = \alpha_1{}^\vee(K^*) \cdots \alpha_\ell{}^\vee(K^*) .$$

<u>Proof</u> : From the structure theorem of finitely generated abelian groups , there exists a $\mathbb{Z}$-basis $(\gamma_1,\ldots,\gamma_\ell)$ of Y and positive integers n_1 , ..., n_r , such that $(n_1\gamma_1,\ldots,n_r\gamma_r)$ is a $\mathbb{Z}$-basis of A . Then we have

$$T = \gamma_1(K^*) \times \cdots \times \gamma_\ell(K^*) ,$$

by Proposition 4.4.1. , and so $A^\wedge = n_1\gamma_1(K^*) \times \cdots \times n_r\gamma_r(K^*)$ as K^* is divisible and n_i are non-zero . Hence the dimension of $A^\wedge$ is r . In particular, as $\alpha_1{}^\vee,\ldots,\alpha_\ell{}^\vee$ are linearly independent , we have rank $<\alpha_1{}^\vee,\ldots,\alpha_\ell{}^\vee> = \ell$, and so $\alpha_1{}^\vee(K^*) \cdots \alpha_\ell{}^\vee(K^*) = T$. ∎

<u>Remark</u> . The decomposition $T = \alpha_1{}^\vee(K^*) \cdots \alpha_\ell{}^\vee(K^*)$ is in general not a direct decomposition of T , as one can see for instance when G has type A_ℓ . (It is enough to take the non-adjoint , non-simply-connected algebraic group of type A_3) .

In the following we shall use the same notation we introduced in the previous paragraph . Therefore , for every α in Φ we have the 1-dimensional subtorus $L_{\alpha^\vee} = \alpha^\vee(K^\times)$ of T , and for every i in I , the surjective algebraic homomorphism

$$\zeta_i : K^\times \longrightarrow L_{\alpha_i^\vee}$$

given by the map $\zeta_i(k) = \alpha_i^\vee(k)$ for every k in $K^\times$. By Proposition 4.5.1. , we have $T = L_{\alpha_1^\vee} \cdots L_{\alpha_\ell^\vee}$. In the previous paragraph we also had the isomorphism

$$\mu_{ij} : K^\times \times K^\times \longrightarrow L_{\alpha_i^\vee} \times L_{\alpha_j^\vee}$$

for every i , j in I such that $\alpha_i+\alpha_j$ lies in Φ . In our case we do not know a priori if the product $L_{\alpha_i^\vee} L_{\alpha_j^\vee}$ is direct . Anyway we can still define

$$\mu_{ij} : K^\times \times K^\times \longrightarrow L_{\alpha_i^\vee} L_{\alpha_j^\vee}$$

by the map $(a,b) \mapsto \zeta_i(a)\zeta_j(b)$ for every a , b in $K^\times$, so that μ_{ij} is a surjective algebraic homomorphism . We shall show that in fact the map μ_{ij} is injective (hence , in particular , $L_{\alpha_i^\vee} L_{\alpha_j^\vee}$ is a direct product) . This will allow us to use again Lemma 4.2.3. to show that every autoprojectivity φ in Γ fixes every subgroup of T generated by a semisimple element of G .

<u>Proposition 4.5.2.</u> Let i , j be elements of I such that $\alpha_i+\alpha_j$ lies in Φ . Then the map μ_{ij} is a bijective algebraic homomorphism . In particular ζ_i is injective for every i in I .

<u>Proof</u> : We already know that μ_{ij} is a surjective algebraic homomorphism . We shall prove that it is injective . Let (a,b) be in $\ker \mu_{ij}$. Then , by the definition of μ_{ij} and of ζ_i , ζ_j , we have $\alpha_i^\vee(a)\alpha_j^\vee(b) = 1$. From the fact that the rank of G is at least 3 , and that the Dynkin diagram of G has only single bonds , there exists k in I , $k \neq i, j$, such that we have one of the following subgraphs

$$\begin{array}{ccccc} \bullet & - & \bullet & - & \bullet \\ k & & i & & j \end{array} \qquad \begin{array}{ccccc} \bullet & - & \bullet & - & \bullet \\ k & & j & & i \end{array}$$

Without loss of generality, we may suppose we are in the first situation. Therefore we have $<\alpha_k, \alpha_i^\vee> = -1$, $<\alpha_k, \alpha_j^\vee> = 0$ and $<\alpha_i, \alpha_j^\vee> = -1$ ([C_2] page 23). We recall that for every r, s in I we have

$$\alpha_r(\alpha_s^\vee(c)) = c^{<\alpha_r, \alpha_s^\vee>} \quad \text{for every } c \text{ in } K^*.$$

We therefore have

$$1 = \alpha_k(1) = \alpha_k(\alpha_i^\vee(a)\alpha_j^\vee(b)) = a^{-1},$$

and so we get $a = 1$, and consequently $\alpha_j^\vee(b) = 1$. But then we have

$$1 = \alpha_i(1) = \alpha_i(\alpha_j^\vee(b)) = b^{-1},$$

which gives $b = 1$. Hence we obtain $(a,b) = (1,1)$, and μ_{ij} is injective. Now let i be in I. As the Dynkin diagram of G is connected, there exists j in I such that $\alpha_i + \alpha_j$ lies in Φ. But then we proved that the map μ_{ij} is injective, and so ζ_i must be injective. #

We now observe that the equation $L_{\alpha^\vee} = T \wedge <X_\alpha, X_{-\alpha}>$ for every α in Φ that we proved in Proposition 4.4.7., holds independently of the isogeny class of G. Hence in our case we still have $L_{\alpha^\vee}{}^\varphi = L_{\alpha^\vee}$ for every α in Φ. We are now able to prove the following

<u>Proposition 4.5.3.</u> Let φ be in Γ, and let T be any maximal torus of G. Then any automorphism of T inducing φ on $L(T)$ is a power-automorphism.

<u>Proof</u>: Let f be an automorphism of T inducing φ on $L(T)$. Let t be in T. By Proposition 4.5.1., for every i in I there exists an element t_i in $L_{\alpha_i^\vee}$ such that $t = t_1 \cdots t_\ell$ (this decomposition being possibly not unique). By Proposition 4.5.2., the homomorphism ζ_i is injective for every i in I, and so, for every i in I, there exists a unique k_i in K^* such that $\zeta_i(k_i) = t_i$. We can now use the same procedure we use in the adjoint and simply-connected case. So let F_{p^α} be the subfield of K generated by $k_1, \ldots, k_\ell$. Let also k be a generator of the multiplicative group of F_{p^α}. For every i in I,

we have $L_{\alpha_i^\vee}^{\varphi} = L_{\alpha_i^\vee}$, and so $L_{\alpha_i^\vee}^{f} = L_{\alpha_i^\vee}$. Therefore , for every i in I , there exists a unique n_i in $\{1,\dots,p^{\alpha}-1\}$ such that $\zeta_i(k)^f = \zeta_i(k^{n_i})$. For every i , j in I such that $\alpha_i+\alpha_j$ lies in Φ , we consider the automorphisms $f_{ij} : L_{\alpha_i^\vee} \times L_{\alpha_j^\vee} \rightarrow L_{\alpha_i^\vee} \times L_{\alpha_j^\vee}$ induced by f , and we denote by V_{ij} the automorphism $\mu_{ij} f_{ij} \mu_{ij}^{-1}$ of $K^{\times} \times K^{\times}$. From the fact that $L_{\alpha_i^\vee}^{f} = L_{\alpha_i^\vee}$, $L_{\alpha_j^\vee}^{f} = L_{\alpha_j^\vee}$ and $L_{\alpha_i^\vee+\alpha_j^\vee}^{f} = L_{\alpha_i^\vee+\alpha_j^\vee}$, it follows that

$$(K^{\times}\times\{1\})^{V_{ij}} = K^{\times}\times\{1\} , \ (\{1\}\times K^{\times})^{V_{ij}} = \{1\}\times K^{\times} \text{ and } D^{V_{ij}} = D$$

(where , as usual , D denotes the diagonal of $K^{\times} \times K^{\times}$. Note that we still have $D^{\mu_{ij}} = L_{\alpha_i^\vee+\alpha_j^\vee}$, as we proved in Proposition 4.4.6. for the simply-connected case) . Hence , by Lemma 4.2.3. , and the connectedness of the Dynkin diagram of G , we get $n_i = n_j$ for every i , j in I . Call this common value n , so that we have

$$\zeta_i(k)^f = \zeta_i(k^n) \text{ for every } i \text{ in } I .$$

We had $t = t_1 \cdots t_\ell$, with t_i in $L_{\alpha_i^\vee}$ for every i in I , and we denoted by k_i the unique element of $K^{\times}$ such that $\zeta_i(k_i) = t_i$. But then , as we showed in the proof of Proposition 4.3.14. , there exist $r_1,\dots,r_\ell$ in $\mathbb{Z}$ such that $k_i = k^{r_i}$ for every i in I . Hence we obtain

$$t = t_1 \cdots t_\ell = \zeta_1(k_1) \cdots \zeta_\ell(k_\ell) = \zeta_1(k^{r_1}) \dots \zeta_\ell(k^{r_\ell}) ,$$

and so

$$t^f = \zeta_1(k^{r_1})^f \dots \zeta_\ell(k^{r_\ell})^f = \zeta_1(k^{nr_1}) \dots \zeta_\ell(k^{nr_\ell}) = t_1^n \cdots t_\ell^n = t^n .$$

Therefore f is a power-automorphism of T . #

Corollary 4.5.4. Let s be a semisimple element of the simple algebraic group G , where G is neither adjoint , nor simply-connected . Then we have

$$\langle s \rangle^{\varphi} = \langle s \rangle$$

for every φ in Γ .

Proof : Let φ be in Γ . There exists a maximal torus T of G such that s lies in T . By Proposition 4.2.2. , 4.5.3. , there exists a power-automorphism f of T inducing φ on $\mathcal{L}(T)$. Therefore we have $\langle s \rangle^{\varphi} = \langle s \rangle^{f} = \langle s \rangle$. #

In the next paragraph we shall summarise the results so far obtained, and we finally deal with unipotent subgroups of G.

§ 4.6 The general case.

In this paragraph we prove that if G is a simple algebraic group over K of rank at least 3 and if the characteristic of K is not 2, then the group Γ coincides with the identity subgroup of Aut $L(G)$, i.e. we have

Aut $L(G)$ = Aut G.

As we noted before, to show that $\Gamma = \{1\}$, it is enough to show that

$<s>^{\varphi} = <s>$ and $<u>^{\varphi} = <u>$

for every φ in Γ, every semisimple element s of G and every unipotent element u of G. We have already proved the following

Proposition 4.6.1. Let s be a semisimple element of the simple algebraic group G. Then, if the rank of G is at least 3, we have

$<s>^{\varphi} = <s>$

for every φ in Γ.

Proof: This follows from Corollary 4.3.15., 4.4.10. and 4.5.4.. #

We are now interested in the behaviour of subgroups generated by a unipotent element under the action of Γ. We shall make use of the classification of unipotent classes of G.

Lemma 4.6.2. Let G be a simple algebraic group over an algebraically closed field. Then every unipotent element u of G is conjugate under G to its inverse u^{-1}.

Proof: We can give a proof from the classification of unipotent conjugacy classes if the characteristic of the field is 0 or a good prime. So let u be a unipotent element of G. From the Bala-Carter theorem ([C_2] Theorem 5.9.6.) and the results of Pommerening ([P_1], [P_2]) (we recall that the classification of unipotent classes is independent of the isogeny class of G), there exists a Levi subgroup L of G and a parabolic subgroup P of the derived subgroup L' of L, such that u lies in the unique conjugacy class C of L' such that $C \cap R_u(P)$ is open and dense in $R_u(P)$ ([C_2] Note on page 132). In the following we shall denote by U the unipotent radical $R_u(P)$ of P. Now we consider the inversion map

$\varepsilon : L' \to L'$ given by $x \mapsto x^{-1}$ for every x in L'.

This map is an automorphism of affine varieties. Let $\{C_1, \ldots, C_k\}$ be the (finite) set of unipotent conjugacy classes of L'. Then ε permutes this set, as v is unipotent $\Leftrightarrow$ v^{-1} is unipotent, and $v \sim v' \Leftrightarrow v^{-1} \sim v'^{-1}$. Now $C \cap U$ is open and dense in U, and so $(C \cap U)^{-1}$ must be open and dense in U, as ε is a homeomorphism of topological spaces, and $U^{-1} = U$. But $(C \cap U)^{-1} = C^{-1} \cap U^{-1}$ $= C^{-1} \cap U$, hence we must have $C^{-1} = C$, as C is the unique conjugacy class C of L' such that $C \cap U$ is open and dense in U, and C^{-1} satisfies this condition as well. Therefore u^{-1} lies in C, which implies $u \sim u^{-1}$ in L'. In particular u and u^{-1} are conjugate in G. If p is a bad prime the classification of unipotent conjugacy classes may be different ([C_2] § 5.11). However it has been shown by J.N. Spaltenstein (private communication) that u is always conjugate to u^{-1} in this case also. ∎

From the previous important result we get the following

Proposition 4.6.3. Let G be a simple algebraic group over K. Then, for every unipotent element u of G, there exists an element g of G such that $gug^{-1} = u^{-1}$ and the order of g is a power of 2.

Proof: Let u be a unipotent element of G. By Lemma 4.6.2., there exists an element h of G such that $huh^{-1} = u^{-1}$. Let $2^a m$ be the order of h, with $2 \nmid m$. Then, if we take $g = h^m$, we have $gug^{-1} = u^{(-1)^m} = u^{-1}$, and the order of g is 2^a. ∎

We can finally prove

Proposition 4.6.4. Let G be a simple algebraic group over K of rank at least 3, and let u be a unipotent element of G. Then, if the characteristic p of K is not 2, we have

$$< u >^{\varphi} = < u >$$

for every φ in Γ.

Proof : By Proposition 4.6.3., there exists an element g in G such that $gug^{-1} = u^{-1}$ and the order of g is a power of 2. As $p \neq 2$, both g and gu are semisimple elements of G (note that $(gu)^2 = g^2$). Therefore, from Proposition 4.6.1., we have $< g >^{\varphi} = < g >$ and $< gu >^{\varphi} = < gu >$ for every φ in Γ. Hence we get

$$< g,u >^{\varphi} = < g, gu >^{\varphi} = < g, gu > = < g,u > \quad \text{for every } \varphi \text{ in } \Gamma .$$

But then we must have $< u >^{\varphi} = < u >$ for every φ in Γ, as $< u >$ is the unique p-Sylow subgroup of the group $< g,u >$, and φ is index-preserving. #

We are now able to prove the announced

Theorem 4.6.5. Let G be a simple algebraic group over the field $K = \bar{F}_p$. Then, if the rank of G is at least 3, and the characteristic of K is not 2, every autoprojectivity of G is induced by a unique automorphism of G.

Proof : We already know that it is enough to prove that the subgroup Γ of $\mathrm{Aut}\, L(G)$ coincides with the identity subgroup of $\mathrm{Aut}\, L(G)$, and furthermore that it is enough to show that $< s >^{\varphi} = < s >$ and $< u >^{\varphi} = < u >$ for every φ in Γ, every semisimple element s of G and every unipotent element u of G. But then this follows from Proposition 4.6.1. and 4.6.4.. #

In the next chapters we shall deal with the case when the rank of G is less than 3.

Chapter 5 The case rank G less than 3 .

In the previous chapter we proved that for simple algebraic groups of rank at least 3, in odd characteristic , every autoprojectivity is induced by an automorphism . We made a crucial use of the hypothesis on the rank to be able to use a theorem by Baer . Here we consider groups of rank 1 or 2 . In this case the above mentioned theorem by Baer drastically fails . Nevertheless we are able to prove the same result we proved for rank $G \geq 3$, for groups of rank 1 and of rank 2 , if we exclude the case when G has type A_2 .

§ 5.1 Simple algebraic groups of rank 1 or 2 .

We give a description of groups of rank 1 or 2 . First we consider the case when G has rank 1 . Then there are just two possibilities :

either G is simply-connected , and in this case G is isomorphic as an algebraic group to the special linear group $SL_2(K)$,

or G is adjoint , and in this case G is isomorphic as an algebraic group to the projective general linear group $PGL_2(K)$.

Note that the algebraic groups $SL_2(K)$ and $PGL_2(K)$ are never isomorphic as algebraic groups , even when the characteristic of K is 2 , i.e. when $SL_2(K)$ and $PGL_2(K)$ are isomorphic as abstract groups .

Suppose now that G has rank 2 . Then G has type A_2 , B_2 or G_2 . If G has type A_2 or B_2 , then there are two possibilities : either G is simply-connected or G is adjoint . We have

$(A_2)_{sc} = SL_3(K)$ and $(A_2)_{ad} = PGL_3(K)$,

while

$(B_2)_{sc}$ is the symplectic group $Sp_4(K)$ and $(B_2)_{ad}$ is the projective conformal symplectic group $PCSp_4(K)$.

Finally , if G is the exceptional group of type G_2 , then there is only one possibility for G, which is both simply-connected and adjoint.

We make the following

Observation 5.1.1. Our aim is to study the group $\mathrm{Aut}\, L(G)$. But clearly our aim will be achieved if we determine the group $\mathrm{Aut}\, L(\bar{G})$ where $\bar{G}$ is any abstract group isomorphic to our given algebraic group G. Now, if G is any simple algebraic group, there exists an isogeny $\pi : G \longrightarrow G_{ad}$, where G_{ad} is the adjoint simple algebraic group of the same type of G. We have $\ker \pi = Z(G)$, and so $G/Z(G)$ is isomorphic to G_{ad} as an abstract group. Note that $G/Z(G)$ has in a natural way the structure of an algebraic group (as a quotient of an algebraic group over a closed normal subgroup). But $G/Z(G)$ and G_{ad} are in general not isomorphic as algebraic groups, as one can see by taking $G = SL_2(K)$, $G_{ad} = PGL_2(K)$ and K of characteristic 2. Nevertheless we have $G/Z(G) \cong G_{ad}$ (as abstract groups) and so every autoprojectivity of $G/Z(G)$ is induced by an automorphism of $G/Z(G)$ if and only if every autoprojectivity of G_{ad} is induced by an automorphism of G_{ad}. #

In view of the previous observation, we introduce the abstract groups

$$PSL_2(K) = SL_2(K)/Z(SL_2(K)) \text{ and } PSp_4(K) = Sp_4(K)/Z(Sp_4(K)).$$

Hence we have

$$PGL_2(K) = (A_1)_{ad} \cong PSL_2(K) \quad \text{and} \quad PCSp_4(K) = (B_2)_{ad} \cong PSp_4(K),$$

as abstract groups.

§ 5.2 The case rank $G = 1$.

In this paragraph we consider groups of rank 1. Therefore either G is isomorphic to $SL_2(K)$ or to $PGL_2(K)$. We shall first determine the group

Aut $L(PSL_2(K))$ and so, by Observation 5.1.1., we shall automatically determine the group Aut $L(PGL_2(K))$. We recall the following result by C. Metelli (cf. [M_1]).

Let $q = p^f$, where p is a prime and f is any natural number. Then, if q is at least 4, for every projectivity τ of the simple group $PSL_2(q)$ onto a group H, there exists a unique isomorphism $\alpha : PSL_2(q) \longrightarrow H$ inducing τ.

We can now prove

Theorem 5.2.1. Let φ be an autoprojectivity of $PSL_2(K)$. Then there exists a unique automorphism of $PSL_2(K)$ inducing φ.

Proof : Let us denote by Z the centre of $SL_2(K)$. For every n in $\mathbb{N}$, we denote by G_n the finite subgroup $SL_2(p^{n!})/Z$ of $PSL_2(K)$ (This matches a definition we gave in chapter 2). Then G_n coincides with $PSL_2(p^{n!})$ for every n in $\mathbb{N}$. We have

$$G_n \leq G_{n+1} \quad \text{for every } n \text{ in } \mathbb{N}\text{, and}$$

$$\bigcup_{n \in \mathbb{N}} G_n = PSL_2(K) \quad .$$

For every n in $\mathbb{N}$, we consider the restrictions

$$\varphi_n : L(G_n) \longrightarrow L(G_n^{\varphi}) \quad \text{induced by } \varphi .$$

Therefore, for every n in $\mathbb{N}$, φ_n is a projectivity of G_n onto G_n^{φ}, and so, by Metelli 's result, for every $n = 2,3,...$, there exists a unique isomorphism

$$\alpha_n : G_n \longrightarrow G_n^{\varphi}$$

inducing φ_n. Let n be in $\mathbb{N}$, $n \geq 2$. We have

$$G_n^{\alpha_{n+1}} = G_n^{\varphi_{n+1}} = G_n^{\varphi} = G_n^{\varphi_n},$$

so we can define $\beta_n : G_n \longrightarrow G_n^{\varphi}$ by the rule $g^{\beta_n} = g^{\alpha_{n+1}}$ for every g in G_n. Then β_n is an isomorphism between G_n and G_n^{φ} inducing φ_n. Therefore we must have $\beta_n = \alpha_n$ by the uniqueness of α_n. Hence we have

$$g^{\alpha_n} = g^{\alpha_{n+1}} \text{ for every } g \text{ in } G_n ,$$

for every n in $\mathbb{N}$, $n \geq 2$. This enables us to define the map

$$\alpha : PSL_2(K) \longrightarrow PSL_2(K)$$

by $g^{\alpha} = g^{\alpha_n}$ for g in $PSL_2(K)$, where n is any natural number greater than 1 such that g lies in G_n . α is clearly an injective homomorphism , and it is also surjective , as $\bigcup_{n \in \mathbb{N}} G_n^{\varphi} = \bigvee_{n \in \mathbb{N}} G_n^{\varphi} = (\bigvee_{n \in \mathbb{N}} G_n)^{\varphi} = PSL_2(K)$.

Hence α is an automorphism of $PSL_2(K)$. Let X be a finite subgroup of $PSL_2(K)$. Then there exists n in $\mathbb{N}$, $n \geq 2$, such that $X \leq G_n$. We have

$$X^{\alpha} = X^{\alpha_n} = X^{\varphi_n} = X^{\varphi} .$$

This is enough to say that α induces φ , as every subgroup of $PSL_2(K)$ is the union of a family of finite subgroups of $PSL_2(K)$.
Uniqueness of α follows from the fact that the group of power-automorphisms of a perfect group is the identity group . #

We have therefore also proved that if G is a simple adjoint algebraic group of rank 1 over K , then every autoprojectivity of G is induced by a unique automorphism of G , i.e.

$$\mathrm{Aut}\, \mathcal{L}(G) = \mathrm{Aut}\, G .$$

We shall now consider the other possibility for groups of rank 1 , i.e. the case when $G = SL_2(K)$. We need some preliminary results .

<u>Proposition 5.2.2.</u> Let G be a connected reductive algebraic group over K . Then we have

$$Z(G)^{\varphi} = Z(G)$$

for every autoprojectivity φ of G .

<u>Proof</u> : The result is trivial if G is abelian . So assume G is not abelian . Let φ be in $\mathrm{Aut}\, \mathcal{L}(G)$. By Proposition 2.2.5. , φ permutes the set of all Borel subgroups of G , and so we get $Z(G)^{\varphi} = Z(G)$, as $Z(G)$ is the intersection of all Borel subgroups of G ([Hu] ex. 2 page 162) . #

<u>Lemma 5.2.3.</u> Let H be the abelian 2-group $\langle y \rangle \times \langle \sigma \rangle$, where y has order 2^{α} with $\alpha \geq 1$, and σ is an involution . Then we have

$\mathrm{Frat}\,(H) = \langle y^2 \rangle$.

Proof : This follows immediately from the fact that the three maximal subgroups of H are $\langle y \rangle$, $\langle y\sigma \rangle$ and $\langle y^2 \rangle \times \langle \sigma \rangle$. ∎

We now prove a lemma of which we shall make use also in the case when G has type B_2 .

Lemma 5.2.4. Let G be one of the groups $SL_2(K)$, $Sp_4(K)$. Suppose φ is an autoprojectivity of G fixing every subgroup of G containing the centre of G . Then φ is the identity .

Proof : Let us denote by Z the centre of G . If the characteristic p of K is 2, then we have $Z = \{1\}$ and so there is nothing to prove . So let $p \neq 2$. Then Z is cyclic of order 2 . To show that $\varphi = 1$, it is enough to show that $\langle x \rangle^{\varphi} = \langle x \rangle$ for every element x of G of prime-power order . So let x be an element of G of order r^{α} , where r is a prime , and $\alpha \geq 1$. If r is odd , then $\langle x \rangle$ is the unique r–Sylow subgroup of $\langle x,Z \rangle$. Hence we must have

$$\langle x \rangle^{\varphi} = \langle x \rangle$$

as , by hypothesis $\langle x,Z \rangle^{\varphi} = \langle x,Z \rangle$, and φ is index-preserving . Now let $r = 2$. We have two cases . If $\langle x \rangle \geq Z$, then we have $\langle x \rangle^{\varphi} = \langle x \rangle$ by hypothesis . Otherwise we have $\langle x \rangle \ngeq Z$. There exists a maximal torus T of G containing x , as x is a semisimple element of G . As T is a divisible group , there exists y in T such that $y^2 = x$. We have $\langle y \rangle \ngeq Z$, as otherwise $\langle x \rangle \geq Z$ (recall that $\alpha \geq 1$) . Hence the group $\langle y,Z \rangle = \langle y \rangle \times Z$ satisfies the hypothesis of Lemma 5.2.3. , and so we get

$$\mathrm{Frat}\,(\langle y,Z \rangle) = \langle y^2 \rangle = \langle x \rangle .$$

But then we have

$$\langle x \rangle^{\varphi} = (\mathrm{Frat}\,(\langle y,Z \rangle))^{\varphi} = \mathrm{Frat}\,(\langle y,Z \rangle^{\varphi}) = \mathrm{Frat}\,(\langle y,Z \rangle) = \langle x \rangle ,$$

as the Frattini subgroup is clearly an invariant under projectivities . ∎

We can now prove the following

Proposition 5.2.5. Let φ be an autoprojectivity of $SL_2(K)$. Then there exists a unique automorphism of $SL_2(K)$ inducing φ.

Proof : Let us denote by Z the centre of $SL_2(K)$. Suppose first that the characteristic p is 2. Then we have $Z = \{1\}$, and so $SL_2(K) = PSL_2(K)$. Hence the result follows from Theorem 5.2.1.. So let us assume $p \neq 2$. By Proposition 5.2.2., we have $Z^{\varphi} = Z$, and so we can define the autoprojectivity $\bar{\varphi}$ of $PSL_2(K)$ by the rule

$$(X/Z)^{\bar{\varphi}} = X^{\varphi}/Z\,,$$

for every subgroup X of $SL_2(K)$ containing Z. By Theorem 5.2.1., there exists an automorphism $\bar{\alpha}$ of $PSL_2(K)$ inducing $\bar{\varphi}$. Also, from the structure of the group Aut $PSL_2(K)$ ([D]), there exists an automorphism α of $SL_2(K)$ inducing $\bar{\alpha}$ on $PSL_2(K)$. We therefore have

$$X^{\varphi} = X^{\alpha} \text{ for every subgroup } X \text{ of } SL_2(K) \text{ containing } Z.$$

Let us denote by ψ the autoprojectivity $\varphi(\alpha^{-1})^{*}$ of $SL_2(K)$. Then we have $X^{\psi} = X$ for every subgroup X of $SL_2(K)$ containing Z, and so, by Lemma 5.2.4., we get $\psi = 1$. Hence we obtain $\varphi = \alpha^{*}$. Uniqueness follows again from the fact that $SL_2(K)$ is a perfect group. #

We can now summarize the previous results in the following

Corollary 5.2.6. Let G be a simple algebraic group of rank 1 over K. Then every autoprojectivity of G is induced by a unique automorphism of G.

Proof : This follows from Theorem 5.2.1., Observation 5.1.1. and Proposition 5.2.5.. #

Therefore, for every simple algebraic group of rank 1 over K, we have Aut $\mathcal{L}(G) =$ Aut G (identifying the groups Aut G and $(\text{Aut } G)^{*}$). In particular we can now say that Corollary 3.4.15. of Chapter 3, holds also when the rank of G is 1 (with $\Gamma = \{1\}$ by Proposition 3.4.10.).

Remark . For simple algebraic groups of rank 1 over K we did not restrict our attention to odd primes (we did so in the case of groups of rank at least 3) .

§ 5.3 The case rank $G = 2$.

We finally consider the case when G has rank 2 . We shall show that if the characteristic p is not 2 , then we still have $\mathrm{Aut}\,\mathcal{L}(G) = \mathrm{Aut}\,G$ if G has type B_2 or G_2 . We shall deal with the case A_2 in the next chapter .

From the results we got in Chapter 3 , we only have to prove that the subgroup Γ of $\mathrm{Aut}\,\mathcal{L}(G)$ coincides with the identity subgroup of $\mathrm{Aut}\,\mathcal{L}(G)$. We recall that Γ is the subgroup of all autoprojectivities of G fixing every parabolic subgroup of G . In this paragraph we shall use the notation $\Gamma(G)$ instead of Γ to specify directly the dependence of Γ on G . We shall make use of the following result , which we prove in a more general context .

Lemma 5.3.1. Let G be a simple simply-connected algebraic group , and let T be a maximal torus of G . Suppose there exists an element n in $\mathcal{N}(T)$ such that

$s^n = s^{-1}$ for every s in T .

Then n^2 lies in the centre of G .

Proof : We first prove that n^2 lies in the centre of $\mathcal{N}(T)$. Let x be any element of $\mathcal{N}(T)$. For every s in T we have

$$s^{x^{-1}nx} = ((s^{x^{-1}})^{-1})^x = (s^{x^{-1}x})^{-1} = s^{-1} = s^n ,$$

and so $x^{-1}nxn^{-1}$ lies in $\mathcal{C}(T) = T$. Therefore we have $n^x = x^{-1}nxn^{-1}n = hn$, where we denote by h the element $x^{-1}nxn^{-1}$ of T . Then we get

$$(n^2)^x = (n^x)^2 = hnhn = nn^{-1}hnhn = nh^{-1}hn = n^2 .$$

Hence n^2 lies in the centre of $\mathcal{N}(T)$. Let us denote by Z the centre of $\mathcal{N}(T)$. We shall now prove that Z lies in $Z(G)$. Let X be the character group $\mathrm{Hom}\,(T, K^*)$ of T . We know that $Z(G)$ coincides with the subgroup $\{ s\in T \mid \alpha(s) = 1\ \forall\alpha\in\Phi \}$ of T (Chapter 1) , where Φ is the set of roots of

G relative to T. We can make W act on X by the rule $(w,x) \mapsto w(x)$, for every w in W and every x in X, where $w(x)$ is the character defined by the map $(w(x))(s) = x(s^w)$ for every s in T (Chapter 1). For every α in Φ let w_α and $\alpha^\vee$ be respectively the reflection and the coroot associated to α. It results

$$w_\alpha(x) = x - \langle x,\alpha^\vee\rangle \alpha \quad \text{for every } x \text{ in } X \text{ ([}C_2\text{] page 19)},$$

where $\langle\ ,\ \rangle : X \times Y \longrightarrow \mathbb{Z}$ is the usual duality between X and the cocharacter group Y of T. As G is simply-connected, for every root α, the coroot $\alpha^\vee$ lies in some $\mathbb{Z}$-basis of Y, and so there exists a character x of T such that $\langle x,\alpha^\vee\rangle = -1$. Therefore we have

$$w_\alpha(x) = x - \langle x,\alpha^\vee\rangle \alpha = x + \alpha,$$

which leaves us with $\alpha = w_\alpha(x) - x$. Hence every root α is of the form

$$\alpha = w(x) - x \quad \text{for some } w \text{ in } W \text{ and } x \text{ in } X.$$

We now note that Z is contained in T, as we have

$$Z = Z(\mathcal{N}(T)) \le C_{\mathcal{N}(T)}(T) \le C_G(T) = T.$$

We can then consider the subgroup $Z^\perp = \{ x \in X \mid x(s) = 1 \ \ \forall s \in Z \}$ of X. Let w be any element of W and let x be any element of X. We prove that $w(x) - x$ lies in $Z^\perp$. So let s be in Z. We have to show that $(w(x) - x)(s) = 1$. But s in Z implies $s^w = s$, and so we get $(w(x) - x)(s) = x(s^w s^{-1}) = x(ss^{-1}) = 1$. It follows that every root α lies in $Z^\perp$ and so we obtain

$$\alpha(s) = 1 \text{ for every } \alpha \text{ in } \Phi \text{ and every } s \text{ in } Z.$$

But this means that every element s of Z lies in $Z(G)$. In particular n^2 lies in $Z(G)$. ∎

Lemma 5.3.2. Let G be an adjoint simple algebraic group whose Weyl group has a non-trivial centre. Then, for every maximal torus T of G, there exists an involution σ of G such that

$$s^\sigma = s^{-1}$$

for every s in T.

Proof : We first make an observation. Let G be any simple algebraic group, and let T be a maximal torus of G. Let $W = \mathcal{N}(T)/T$ be the Weyl group and X be the character group of T. The centre of W has order 1 or 2. Suppose we are in the second case. Then the non-trivial element of $Z(W)$ is the longest element w_0 of W, and we have $w_0(x) = -x$ for every x in X, under the same action of W on X we considered before ([Bo] corollary on page 146). Therefore we get

$(w_0(x))(s) = x(s^{-1})$ for every s in T and every x in X.

But by the definition of $w_0(x)$, we have $(w_0(x))(s) = x(s^{w_0})$ and so we obtain $x(s^{w_0} s) = 1$ for every s in T and every x in X. This implies that $s^{w_0} s = 1$ for every s in T, as the identity is the unique element of T fixed by every character of T. Hence we have

$s^{w_0} = s^{-1}$ for every s in T.

We now go back to the proof of the proposition. Let G_1 be the simply-connected covering of G, and let $\pi : G_1 \rightarrow G$ be the associated isogeny. Then we have $\ker \pi = Z(G_1)$, as G is adjoint. Let T be a maximal torus of G, and let us denote by T_1 the maximal torus $\pi^{-1}(T)$ of G_1. The Weyl group $W_1 = \mathcal{N}(T_1)/T_1$ is isomorphic to the Weyl group $W = \mathcal{N}(T)/T$ of G, and so its centre is non-trivial. By the previous observation, we have $s^{w_0} = s^{-1}$ for every s in T_1, where w_0 is the longest element of W_1. Let n be any representative of w_0 in $\mathcal{N}(T_1)$. Then we get $s^n = s^{-1}$ for every s in T_1, from which it follows also that n^2 lies in $Z(G_1)$ by Lemma 5.3.1.. If now we take σ to be the image of n under π, we get

$s^{\sigma} = s^{-1}$ for every s in T,

and $\sigma^2 = 1$ as $Z(G_1) = \ker \pi$. #

We start now to prove that $\Gamma(G)$ is the identity by proving the following

<u>Proposition 5.3.3.</u> Let G be an adjoint simple algebraic group of rank 2 over K. If the characteristic of K is not 2, then for every φ in $\Gamma(G)$ we have

$\langle\sigma\rangle^{\varphi} = \langle\sigma\rangle$

for every involution σ of G.

Proof : Let φ be in $\Gamma(G)$, and let σ be any involution of G . As $p \neq 2$, σ is semisimple , and so there exists a maximal torus T of G containing σ . Let $\Pi = \{\alpha_1, \alpha_2\}$ be a fundamental system for the set of roots of G relative to the choice of T . With the notation we introduced in § 4.3 , we have

$$T = T_{\alpha_1} \times T_{\alpha_2} .$$

We recall that T_{α_i} is a one-dimensional subtorus of T for every $i = 1, 2$. Therefore , being isomorphic to K^* , T_{α_i} has a unique involution τ_i for every $i = 1, 2$. By Proposition 4.3.12. , φ fixes both T_{α_1} and T_{α_2} , and so we must also have $<\tau_1>^\varphi = <\tau_1>$ and $<\tau_2>^\varphi = <\tau_2>$ as φ is index-preserving . Then we get $<\tau_1,\tau_2>^\varphi = <\tau_1,\tau_2>$, from which it follows $<\tau_1\tau_2>^\varphi = <\tau_1\tau_2>$. But then we conclude by observing that σ must coincide with τ_1 or τ_2 or $\tau_1\tau_2$. #

Proposition 5.3.4. Let G be an adjoint simple algebraic group of type B_2 or G_2 over the field K of characteristic not 2 . Let φ be in $\Gamma(G)$. Then we have

$$<s>^\varphi = <s>$$

for every semisimple element s of G .

Proof : Let s be a semisimple element of G , and let T be a maximal torus of G containing s . As G has type B_2 or G_2 , the centre of the Weyl group of G is non-trivial . Therefore , by Lemma 5.3.2. , there exists an involution σ in G such that $t^\sigma = t^{-1}$ for every t in T . In particular we have $s^\sigma = s^{-1}$. We consider the dihedral group $<s,\sigma>$. Being generated by involutions , this group is therefore fixed by φ , by Proposition 5.3.3. . If now s has order 2 , then we conclude by Proposition 5.3.3. . Otherwise $<s>$ is the unique maximal cyclic subgroup of $<s,\sigma>$, and therefore it must be fixed by φ , as φ fixes $<s,\sigma>$. #

For the behaviour of unipotent subgroups under the action of $\Gamma(G)$, we can use the result we proved in Chapter 4 concerning the classification of unipotent conjugacy classes . We therefore have the following

Proposition 5.3.5. Let G be an adjoint simple algebraic group of type B_2 or G_2 over the field K of characteristic not 2. Let φ be in $\Gamma(G)$. Then we have

$$\langle u\rangle^{\varphi} = \langle u\rangle$$

for every unipotent element u of G.

Proof : We can use the same proof of Proposition 4.6.4.. #

We are now able to state

Theorem 5.3.6. Let G be an adjoint simple algebraic group of type B_2 or G_2 over the field K of characteristic not 2. Then every autoprojectivity of G is induced by a unique automorphism of G.

Proof : We know that this is equivalent to prove that $\Gamma(G) = \{1\}$. This then follows from Proposition 5.3.4. and 5.3.5.. #

Let us now consider the simply-connected case. As we already observe that the simply-connected simple algebraic group of type G_2 is also adjoint, we only have to deal with the simply-connected group of type B_2, i.e. with the symplectic group $Sp_4(K)$.

Proposition 5.3.7. Let G be the group $Sp_4(K)$ over the field K of characteristic not 2. Then every autoprojectivity of G is induced by a unique automorphism of G.

Proof : It is enough to show that $\Gamma(G) = \{1\}$. So let φ be in $\Gamma(G)$. We consider the isogeny $\pi : G \longrightarrow G_{ad}$. Then π induces an abstract isomorphism μ between $G/Z(G)$ and G_{ad} (see also Observation 5.1.1.). We also recall that π induces a bijection between the set of parabolic subgroups of G and the set of parabolic subgroups of G_{ad} (Corollary 21.3C and Proposition B on page 148 in [Hu]). By Proposition 5.2.2., we have $Z(G)^{\varphi} = Z(G)$, and so we can define the autoprojectivity ϕ of $G/Z(G)$ by the rule

$$(X/Z(G))^{\phi} = X^{\varphi}/Z(G),$$

for every subgroup X of G containing $Z(G)$. If we denote by γ the projectivity of $G/Z(G)$ onto G_{ad} induced by the isomorphism μ, then we can define the autoprojectivity ψ of G_{ad} by the map $\psi = \gamma^{-1}\phi\gamma$. From the fact that φ lies in $\Gamma(G)$, it follows that ψ lies in $\Gamma(G_{ad})$. But then we must have $\psi = 1$, by Theorem 5.3.6., and so we are left with $\phi = 1$. Hence we get

$$(X/Z(G))^{\phi} = X/Z(G) ,$$

for every subgroup X of G containing $Z(G)$, and so we have $X^{\varphi} = X$ for every subgroup X of G containing $Z(G)$. But then we must have $\varphi = 1$, by Lemma 5.2.4.. #

We can now summarize the previous results in the following

<u>Corollary 5.3.8.</u> Let G be a simple algebraic group of type B_2 or G_2 over the field K of characteristic not 2. Then every autoprojectivity of G is induced by a unique automorphism of G.

<u>Proof</u> : This follows from Theorem 5.3.6. , Observation 5.1.1. and Proposition 5.3.7.. #

In the next chapter we shall deal with the case left out in the discussion so far developed, i.e. the case when G has type A_2.

Chapter 6 The case when G has type A_2.

In the previous chapters we proved that if G is a simple algebraic group over K, where K has odd characteristic, then every autoprojectivity of G is induced by a unique automorphism of G if G is not of type A_2. In this chapter we shall prove that the case when G has type A_2 is indeed an exceptional one. In fact we shall show that for the group $G = SL_3(\bar{F}_{23})$, the corresponding group Γ of exceptional autoprojectivities of G is non-trivial. Therefore every non-trivial element of Γ will be an autoprojectivity of G not induced by any automorphism of G.

We shall make use of a result obtained by H. Völklein for the finite groups $SL_3(q)$ in [V_1]. Unfortunately we found a mistake in the proof of this result, and in fact we could exhibit a counterexample showing that Proposition 3 and the following corollary in [V_1] are not correct. Nevertheless we were able, by modifying the assumptions, to show that the thesis stated in the above mentioned proposition is still valid ([Cs_3]). In the next paragraph we shall construct the above mentioned counterexample.

§ 6.1 The group $SL_3(27)$ is strongly lattice determined.

First we give some notation that we shall use in this and in the next paragraph. By G we shall denote the group $SL_3(q)$ unless otherwise specified, where q is a power of a prime p. We shall denote by U the group of upper unitriangular matrices of G, and by T the group of diagonal matrices of G. We also define the following three subgroups of U by

$$U_1 = \left\{ \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid k \in F_q \right\}, \qquad U_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \mid k \in F_q \right\}$$

$$U_3 = \left\{ \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid k \in F_q \right\}$$

U_1, U_2 and U_3 are the usual root-subgroups of G.

We recall Proposition 3 and the following corollary in $[V_1]$.

<u>Proposition 3</u>. Let G be one of the groups $SL_3(q)$, $SU_3(q)$, and let λ be an autoprojectivity of the group T of diagonal matrices of G. Then λ can be extended to an autoprojectivity of G fixing every p-subgroup of G and commuting with the inner automorphisms of G, if the following holds :

(i). λ commutes with the action of $\mathcal{N}(T)/T$;

(ii). λ fixes every subgroup of T which is fixed by a non-trivial element of $\mathcal{N}(T)/T$;

(iii). λ fixes every 2-subgroup and every 3-subgroup of T;

(iv). λ fixes every subgroup of T which does not act irreducibly on each of the groups U_i for $i = 1,2,3$.

<u>Corollary.</u> The group Γ of exceptional autoprojectivities of $SL_3(p^f)$ is not trivial if there exists a prime $\ell \geq 11$ such that $p^f \equiv 1 \mod \ell$, $p^f \not\equiv 1 \mod \ell^2$ and $p^s \not\equiv 1 \mod \ell$ for every s such that $0 < s < f$.

We shall show that the group $SL_3(27)$ is strongly lattice determined, so that the corollary is not correct as one can see by taking $p = 3$, $f = 3$ and the prime ℓ to be equal 13. Also we shall construct a family of non-trivial autoprojectivities of T satisfying the hypothesis of Proposition 3, but which of course cannot be extended to autoprojectivities of the whole G fixing every 3-subgroup of G (and commuting with the inner automorphisms of G).

So let $G = SL_3(27)$. G is a simple group, and so we already know that to show that G is strongly lattice determined, it is enough to show that every autoprojectivity of G is induced by an automorphism (Chapter 1). We define Γ to be the group of all autoprojectivities of G fixing every 3-Sylow subgroup of G. This matches the definition we gave in Chapter 3. Identifying

Aut G with the subgroup of autoprojectivities of G induced by automorphisms , we have , by Proposition 2 in [V_1] ,

$$\mathrm{Aut}\, \mathcal{L}(G) = \Gamma \rtimes \mathrm{Aut}\, G \ .$$

To show that G is strongly lattice determined , it will be enough to prove that Γ is the identity subgroup of Aut $\mathcal{L}(G)$.

From the corollary on page 11 of [V_2] , to prove that $\Gamma = \{1\}$ we only need to show that $X^\varphi = X$ for every subgroup X of T and every φ in Γ . In our case T is isomorphic to $C_{26} \times C_{26}$, and so , by Lemma 1 in [V_2] , it is enough to show that every φ in Γ fixes every subgroup of order 13 of T . Let $\mathcal{M}$ be the set of the 14 subgroups of order 13 of T . If we make the Weyl group $W = \mathcal{N}(T)/T$ act naturally on $\mathcal{M}$, we get four orbits , of which two have three elements , one has two elements and one has six elements. Let us call this last orbit $\mathcal{E}$. We now fix an element φ of Γ . As we showed in [Cs_2] , φ fixes every subgroup of T which lies in the orbits with two or three elements . To prove that the same holds for the elements of $\mathcal{E}$, we need a description of these subgroups . Let u be a fixed element of order 13 in $F_{27}^{\times}$, let e be the element $\mathrm{diag}(u, u^{10}, u^2)$ of T , and $E = \langle e \rangle$. Then E lies in $\mathcal{E}$ and for every E' in $\mathcal{E}$ there exists a unique w in W such that $E' = E^w$. Let now P be the following subgroup of the group of upper-unitriangular matrices

$$P = \left\{ \begin{pmatrix} 1 & k & k^3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ \middle| \ k \in F_{27} \right\}$$

From a direct calculation , it follows that E is contained in $\mathcal{N}(P)$, and even that E is the unique subgroup in $\mathcal{E}$ satisfying this condition . Also, we observe that E^φ lies in $\mathcal{E}$, as φ is index preserving and we already know that it fixes all the subgroups of order 13 of T not in $\mathcal{E}$. Now P is the unique 3-Sylow subgroup of the group EP , thus P^φ is the unique 3-Sylow subgroup of $\langle E^\varphi , P^\varphi \rangle$. Hence we get

$E^{\varphi} \le \mathcal{N}(P^{\varphi})$.

Finally, as by Proposition 1 in $[V_2]$ P is fixed by φ, we are left with $E^{\varphi} \le \mathcal{N}(P)$, which gives $E^{\varphi} = E$, since E is the unique subgroup in $\mathcal{E}$ contained in $\mathcal{N}(P)$. If now E' is any element of $\mathcal{E}$, we just need to take any g in $\mathcal{N}(T)$ such that $E' = E^g$, and apply the previous argument to the groups E' and P^g. Thus Γ is the identity group, and G is strongly lattice determined.

Now we shall construct for every w in W an autoprojectivity φ_w of T satisfying the conditions of Proposition 3 in $[V_1]$. So let w be a fixed element of W. We define φ_w to be the identity on the 2-subgroups of T and on the subgroups of order 13 which lie in orbits of length two or three of $\mathcal{M}$. If E' is any element in $\mathcal{E}$, there exists a unique ρ in W such that $E' = E^{\rho}$. We then put

$E'^{\varphi_w} = E^{w\rho}$.

There exists a unique way to extend φ_w to an autoprojectivity of T. From a direct calculation it is possible to show that, for every w in W, φ_w satisfies the conditions (i).-(iii). of Proposition 3 (and also that for every φ in Aut $L(T)$ satisfying (i).-(iii). of Proposition 3 there exists a unique w in W such that $\varphi = \varphi_w$). From the fact that 13 does not divide 3^r-1 for $r = 1$ or 2, it follows that φ_w satisfies also condition (iv).. Therefore we can say that for every non trivial w in W we have a non trivial autoprojectivity φ_w of T which satisfies the hypothesis of Proposition 3, but which does not fit with the thesis, because we already know that $\Gamma = \{1\}$. Besides, by taking the prime ℓ equal 13, we see that G represents a counterexample also for the corollary following Proposition 3 in $[V_1]$.

The point is that if λ satisfies the hypothesis of Proposition 3, X is a subgroup of T not fixed by λ and P is a p-subgroup of $SL(3,p^n)$ such that $X \le \mathcal{N}(P)$, then condition (iv). is not enough to guarantee that $X^{\lambda} \le \mathcal{N}(P)$ (step 3 in the proof of Proposition 3). In the following paragraph we are going to modify the content of condition (iv)..

§ 6.2 A procedure for extending autoprojectivities of the group of diagonal matrices of $SL_3(q)$ to the whole of $\mathcal{L}(SL_3(q))$.

We keep the same notation we introduced in the previous paragraph, so that G will always denote the group $SL_3(q)$. Also, just for this paragraph we shall denote by K the field F_q. If F is a subfield of K, we denote by $T(F)$ the subgroup of T whose elements have entries in F.

Let $s = \mathrm{diag}(\lambda, \mu, \nu)$ be an element of T. Then we put :

$$\alpha_1(s) = \lambda\mu^{-1} , \quad \alpha_2(s) = \mu\nu^{-1} , \quad \alpha_3(s) = \lambda\nu^{-1} \ (= \alpha_1(s)\alpha_2(s)) .$$

We shall just write α_i for $\alpha_i(s)$ when there is no ambiguity. Also, for $i = 1, 2, 3$, we denote by $\mu_i(s)$ the minimum polynimial of $\alpha_i(s)$ over F_p. Again we will just write μ_i for $\mu_i(s)$ when there is no ambiguity.

<u>Definition 6.2.1.</u> We say that an element s in T satisfies $(*)$ if

(i). $|s| = |\alpha_i(s)|$ for every $i = 1, 2, 3$; (which implies that $\deg \mu_i = \deg \mu_j$ for every $i, j = 1, 2, 3$) ;

(ii). $\mu_i \neq \mu_j$ for every $i \neq j$.

We say that an element s in T satisfies $(**)$ if nsn^{-1} satisfies $(*)$ for every n in $\mathcal{N}(T)$. #

Suppose that s is an element of T satisfying $(*)$. Then we have

$F_p(\alpha_1) = F_p(\alpha_2) = F_p(\alpha_3) = F_{p^n}$,

where n is the degree of the μ_i's. We shall denote this subfield of K by $F(s)$, and simply by F if there is no ambiguity.

We shall aim to show that for any p-subgroup P of G, and every element s of T satisfying (**) we have $T(F(s)) \leq \mathcal{N}(P)$. The next few propositions will be devoted to a proof of this fact .

<u>Proposition 6.2.2.</u> Let s be an element of T satisfying (*) . Then the map

$$\Psi : F_p[X] \to F \times F \times F$$

given by

$$\theta \mapsto (\theta(\alpha_1), \theta(\alpha_2), \theta(\alpha_3))$$

is a surjective F_p– algebra homomorphism .

<u>Proof</u> : It is clear that Ψ is an F_p– algebra homomorphism . To show that Ψ is surjective we observe that condition (*) implies that $\ker \Psi = (\mu_1 \mu_2 \mu_3)$. Hence we have an induced F_p– algebra monomorphism

$$\bar{\Psi} : F_p[X] / (\mu_1 \mu_2 \mu_3) \to F \times F \times F .$$

But now $F_p[X] / (\mu_1\mu_2\mu_3)$ and $F \times F \times F$ are both F_p–spaces of dimension $3n$, where n is $\deg \mu_i$. Hence $\bar{\Psi}$ is an isomorphism , and Ψ is surjective .#

We now consider the three monomorphisms $x_i : K \to U$ defined by

$$x_1(k) = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad x_2(k) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \qquad x_3(k) = \begin{pmatrix} 1 & 0 & -k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for every k in K .

We therefore have the commutator formula

$$x_2(b)x_1(a) = x_1(a)x_2(b)x_3(ab) \qquad \text{for every } a, b \text{ in } K .$$

Also , from the fact that $\mathrm{Im}\, x_i$, for $i = 1, 2, 3$, are exactly the three root–subgroups of G relative to T contained in U , for every u in U there exists a unique 3–tuple (a, b, c) with a, b, c in K such that

$$u = x_1(a)x_2(b)x_3(c) .$$

In the following we fix an element u in U and hence three elements a, b, c of K such that $u = x_1(a)x_2(b)x_3(c)$.

Proposition 6.2.3. For every n in $\mathbb{Z}$ there exists $h(n)$ in $\mathbb{Z}$ such that

$$u^n = x_1(na)x_2(nb)x_3(nc+h(n)ab) .$$

Proof : We can take $h(n) = 0$ if $n = 0, 1$; $h(n) = n(n+1)/2$ if $n \geq 2$ and $h(n) = (-n+1)(-n+2)/2$ if $n < 0$. The result then comes by induction . $\blacksquare$

We now fix s in T such that s satisfies $(*)$.

Proposition 6.2.4. Let θ be an element of $\mathbb{F}_p[X]$. Then there exists γ in F such that

$$x_1(\theta(\alpha_1)a)x_2(\theta(\alpha_2)b)x_3(\theta(\alpha_3)c+\gamma ab)$$

lies in $\langle u\rangle^{\langle s\rangle}$.

Proof : We use induction on $\deg\theta$. Suppose $\deg\theta = 0$, then $\theta = k$ lies in $\mathbb{F}_p$. Choose n in $\mathbb{Z}$ such that $n \mapsto k$ under the natural map $\pi : \mathbb{Z} \to K$ given by $m \mapsto m.1_K$ for every m in $\mathbb{Z}$. Then , from Proposition 6.2.3. , there exists $h(n)$ in $\mathbb{Z}$ such that

$$u^n = x_1(na)x_2(nb)x_3(nc+h(n)ab) .$$

But then $x_1(ka)x_2(kb)x_3(kc+(h(n).1_K)ab) = u^n$ lies in $\langle u\rangle^{\langle s\rangle}$, $k = \theta(\alpha_i)$ for every

$i = 1, 2, 3$ and $h(n).1_K$ is in F . So now assume the result for all θ in $\mathbb{F}_p$ such that $\deg\theta \leq r$ and let θ be of degree $r+1$. Then we have

$$\theta = \theta' + k_{r+1}X^{r+1} ,$$

where θ' has degree $\leq r$, and k_{r+1} is in $\mathbb{F}_p$. Then , by induction , there exists γ' in F such that

$$v = x_1(\theta'(\alpha_1)a)x_2(\theta'(\alpha_2)b)x_3(\theta'(\alpha_3)c+\gamma' ab)$$

lies in $\langle u\rangle^{\langle s\rangle}$. Choose n_{r+1} in $\mathbb{Z}$ such that $\pi(n_{r+1}) = k_{r+1}$, and let

$$w = (s^{r+1}us^{-(r+1)})^{n_{r+1}} .$$

Then w lies in $\langle u\rangle^{\langle s\rangle}$. Also we have

$$w = x_1(n_{r+1}\alpha_1^{r+1}a)x_2(n_{r+1}\alpha_2^{r+1}b)x_3(n_{r+1}\alpha_3^{r+1}c+h(n_{r+1})\alpha_3^{r+1}ab)$$

as $\alpha_3 = \alpha_1\alpha_2$.

Let $\gamma'' = h(n_{r+1})\alpha_3^{r+1}$, so that γ'' is in F . Consider now the element vw, which is in $\langle u \rangle^{\langle \Phi \rangle}$. Using the commutator formula , we have

$$vw = x_1((\theta'(\alpha_1)+n_{r+1}x_1^{r+1})a)x_2((\theta'(\alpha_2)+n_{r+1}\alpha_2^{r+1})b).$$
$$.x_3((\theta'(\alpha_3)+n_{r+1}\alpha_3^{r+1})c+(\gamma'+\gamma''+n_{r+1}\alpha_1^{r+1}\theta'(\alpha_2))ab) ,$$

which gives the result we want , by defining

$$\gamma = \gamma' + \gamma'' + n_{r+1}\alpha_1^{r+1}\theta'(\alpha_2)$$

and noting that

$$\theta'(\alpha_i) + n_{r+1}\alpha_i^{r+1} = \theta(\alpha_i)$$

for every $i = 1, 2, 3$. #

Proposition 6.2.5. For every A , B , C in F there exists k in F such that

$$x_1(Aa)x_2(Bb)x_3(Cc+kab)$$

lies in $\langle u \rangle^{\langle \Phi \rangle}$.

Proof : By Proposition 6.2.2. , there exists θ in $F_p[X]$ such that

$$\theta(\alpha_1) = A , \theta(\alpha_2) = B \text{ and } \theta(\alpha_3) = C .$$

Then , by Proposition 6.2.4. , there exists γ in F such that

$$x_1(\theta(\alpha_1)a)x_2(\theta(\alpha_2)b)x_3(\theta(\alpha_3)c+\gamma ab)$$

lies in $\langle u \rangle^{\langle \Phi \rangle}$. So we just need to take $k = \gamma$ to get the result . #

Proposition 6.2.6. Let D be in F . Then

$$x_3(Dab)$$

lies in $\langle u \rangle^{\langle \Phi \rangle}$.

Proof : Suppose we have ξ_1 , ξ_2 , ζ , ζ' in K and let

$$y = x_1(\xi_1)x_2(\xi_2)x_3(\zeta) \quad , \quad y' = x_1(\xi_1)x_2(\xi_2)x_3(\zeta') \ .$$

Then we have

$$yy'^{-1} = x_1(\xi_1)x_2(\xi_2)x_3(\zeta)\, x_3(-\zeta')x_2(-\xi_2)x_1(-\xi_1) = x_3(\zeta-\zeta')$$

as $\operatorname{Im} \chi_3 = Z(U)$.

We apply this to the following situation .

Let A , A' , B , B' be elements of F . From Proposition 6.2.5. there exist k , k' in F such that

$$v = x_1(Aa)x_2(Bb)x_3(kab) \qquad \text{and} \qquad w = x_1(A'a)x_2(B'b)x_3(k'ab)$$

are both in $\langle u\rangle^{\langle s\rangle}$. Then vw and wv are both in $\langle u\rangle^{\langle s\rangle}$, and we have

$$vw = x_1((A+A')a)x_2((B+B')b)x_3((k+k'+A'B)ab) \ ,$$

$$wv = x_1((A+A')a)x_2((B+B')b)x_3((k+k'+AB')ab) \ .$$

Hence $x_3((A'B-AB')ab) = (vw)(wv)^{-1}$ lies in $\langle u\rangle^{\langle s\rangle}$.

Finally, if we let $A' = D$, $B = 1$, $A = B' = 0$, then we obtain that $x_3(Dab)$ lies in $\langle u\rangle^{\langle s\rangle}$, as we wanted. ∎

Proposition 6.2.7. Let A, B, C, D be elements of F. Then

$$x_1(Aa)x_2(Bb)x_3(Cc+Dab)$$

lies in $\langle u\rangle^{\langle s\rangle}$.

Proof : From Proposition 6.2.5., there exists k in F such that

$$v = x_1(Aa)x_2(Bb)x_3(Cc+kab)$$

lies in $\langle u\rangle^{\langle s\rangle}$. Then take $k' = D - k$ and apply Proposition 6.2.6. to obtain that

$w = x_3(k'ab)$ is in $\langle u\rangle^{\langle s\rangle}$. Hence

$$x_1(Aa)x_2(Bb)x_3(Cc+Dab) = vw$$

lies in $\langle u\rangle^{\langle s\rangle}$. ∎

We are now able to state the result that we shall use in the final step.

Lemma 6.2.8. Let s be an element of T satisfying $(*)$. If P is a subgroup of U such that s lies in $\mathcal{N}(P)$, then $T(F(s))$ is contained in $\mathcal{N}(P)$.

Proof : Let t be in $T(F(s))$, i.e. $t = \mathrm{diag}(\lambda', \mu', \nu')$, with λ', μ', ν' in $F(s)$. Then, if $u = x_1(a)x_2(b)x_3(c)$ lies in P, we have

$$tut^{-1} = x_1(\lambda'\mu'^{-1}a)x_2(\mu'\nu'^{-1}b)x_3(\lambda'\nu'^{-1}c) \ ,$$

which is in $\langle u\rangle^{\langle s\rangle}$ by Proposition 6.2.7.. Hence, for every u in P and every t in $T(F(s))$, we have

$$tut^{-1} \in \langle u\rangle^{\langle s\rangle} \leq P^{\langle s\rangle} = P \ .$$

So for every t in $T(F(s))$ we have $P^t \leq P$, which implies that $T(F(s))$ is contained in $\mathcal{N}(P)$. ∎

We are now in the position to prove the following statement, which represents the announced modification of Proposition 3 in $[V_1]$

Proposition 6.2.9. Let G be one of the groups $SL_3(q)$, $SU_3(q)$ where q is a power of a prime p. Let λ be an autoprojectivity of the group T of diagonal matrices of G. Then λ can be extended to an autoprojectivity of G fixing every p-subgroup of G and commuting with the inner automorphisms of G, if the following holds

(i). λ commutes with the action of $\mathcal{N}(T)/T$;

(ii). λ fixes every subgroup of T which is fixed by a non-trivial element of $\mathcal{N}(T)/T$;

(iii). λ fixes every 2-subgroup and every 3-subgroup of T;

(iv'). if s is a prime power order element of T not satisfying $(**)$, then λ fixes the subgroup generated by s.

Proof : We shall prove that if P is a p-subgroup of G normalized by a subgroup X of T, then also X^{λ} and $X^{\lambda^{-1}}$ normalize P. It is then possible to follow the proof of Proposition 3 in $[V_1]$ from step (5) to get the result. So let X be a subgroup of T and P be a p-subgroup of G normalized by X. If X is fixed by λ, then there is nothing to prove. So assume that λ does not fix X. Without loss of generality we may assume that X is cyclic of prime power order. By condition (iv')., there exists a generator s of X satisfying $(**)$. We can apply the same argument of step (4) in the proof of Proposition 3 in $[V_1]$, to obtain an element n in $\mathcal{N}(T)$ such that nPn^{-1} is contained in U. Now nsn^{-1} satisfies $(*)$, and so we can apply the Lemma 6.2.8. to get

$$T(F(nsn^{-1})) \leq \mathcal{N}(nPn^{-1}).$$

Then we have that $nX^{\lambda}n^{-1}$ and $nX^{\lambda^{-1}}n^{-1}$ are both contained in $n\mathcal{N}(P)n^{-1}$, as $T(F(nsn^{-1}))$ contains every subgroup of order $|s|$ of T, and λ is index-preserving. Hence we have that both X^{λ} and $X^{\lambda^{-1}}$ are contained in $\mathcal{N}(P)$, and we are done. ∎

Our aim will be to extend certain autoprojectivities of the subgroup of diagonal matrices of the group $SL_3(\bar{F}_{23})$ to the whole of $\mathcal{L}(SL_3(\bar{F}_{23}))$. In the next paragraph we begin by considering more closely the group of autoprojectivities of 2-dimensional tori .

§ 6.3 Autoprojectivities of 2-dimensional tori

We are interested in the group of autoprojectivities of 2-dimensional tori . The reason for this is that in the next paragraph we shall extend certain autoprojectivities of the group of diagonal matrices (which is a 2-dimensional torus) of $SL_3(\bar{F}_{23})$ to the whole of $\mathcal{L}(SL_3(\bar{F}_{23}))$.

So let T be a 2-dimensional torus . Then T is isomorphic to the direct product of 2 copies of $K^\times$. From § 4.2 we have

$T_r \cong C_{r^\infty} \times C_{r^\infty}$ if r is a prime , $r \neq p$, and

$T_p = \{1\}$.

We also recall from Proposition 4.2.1. that every autoprojectivity of T is index-preserving . Hence , for every φ in $\mathrm{Aut}\,\mathcal{L}(T)$ and every prime r , we shall have

$T_r^{\varphi} = T_r$.

As the group of autoprojectivities of the group $\mathrm{Dr}\,(C_{r^\infty} \times C_{r^\infty})$ (r prime , $r \neq p$) is the cartesian product of the groups $\mathrm{Aut}\,\mathcal{L}(C_{r^\infty} \times C_{r^\infty})$ (r prime , $r \neq p$) , we are therefore left with the problem to study the group of autoprojectivities of the group $C_{r^\infty} \times C_{r^\infty}$ where r is a prime .

For this purpose we introduce the concept of the meet-semilattice of a group .

<u>Definition 6.3.1.</u> For every group H , we define $\mathcal{SL}(H)$ to be the set of all cyclic subgroups of H . #

$\mathcal{SL}(H)$ is a subset of the lattice $\mathcal{L}(H)$ with the property that for every A , B in $SL(\mathrm{H})$, then $A \wedge B$ lies in $SL(\mathrm{H})$. Such a set is called a meet-semilattice (for a more general definition see [Bi] page 22). A bijection φ of $SL(\mathrm{H})$ onto itself will be called an *automorphism* of $SL(\mathrm{H})$ if the following holds

$A \leq B$ if and only if $A^{\varphi} \leq B^{\varphi}$ for every A, B in $SL(\mathrm{H})$.

If φ is an autoprojectivity of H, then clearly the restriction of φ to $SL(\mathrm{H})$ is an automorphism of $SL(\mathrm{H})$. We are interested in the converse, i.e. we would like to know whether an automorphism of $SL(\mathrm{H})$ can be extended (uniquely) to an autoprojectivity of H. This is not in general true, as one can see from the following example. Let H be the group $C_r \times C_r \times C_r$. Then every permutation of $SL(\mathrm{H})$ fixing the identity subgroup is an automorphism of $SL(\mathrm{H})$, but clearly the automorphism φ interchanging the cyclic subgroups $C_r \times \{1\} \times \{1\}$ and $\{1\} \times \{1\} \times C_r$, and fixing all the other elements of $SL(\mathrm{H})$ cannot be extended to an autoprojectivity of H. However we shall show that for the groups $C_{r^\infty} \times C_{r^\infty}$ every automorphism of $SL(C_{r^\infty} \times C_{r^\infty})$ can be extended (uniquely) to an autoprojectivity of $C_{r^\infty} \times C_{r^\infty}$.

So let r be a fixed prime. We denote by R the group $C_{r^\infty} \times C_{r^\infty}$.

<u>Observation 6.3.2.</u> Every finite subgroup M of R is of the form $M = A \times B$, where A, B are in $SL(\mathrm{R})$, $A \cong C_{r^\alpha}$, $B \cong C_{r^\beta}$ and $\alpha \geq \beta$. Also, for every α in $\mathbb{N}$, there exists a unique subgroup R_α of R isomorphic to $C_{r^\alpha} \times C_{r^\alpha}$ (namely $R_\alpha = \Omega_\alpha(R) = \{ x \in R \mid x^{r^\alpha} = 1 \}$). ∎

In the following φ will always denote an automorphism of $SL(\mathrm{R})$. Note that φ is index-preserving.

<u>Proposition 6.3.3.</u> Let C, D be in $SL(\mathrm{R})$. Then we have

$(C \wedge D)^{\varphi} = C^{\varphi} \wedge D^{\varphi}$.

<u>Proof</u>: From $C \wedge D \leq C, D$ we get $(C \wedge D)^{\varphi} \leq C^{\varphi}, D^{\varphi}$ and so $(C \wedge D)^{\varphi} \leq C^{\varphi} \wedge D^{\varphi}$. Similarly we have $(C^{\varphi} \wedge D^{\varphi})^{\varphi^{-1}} \leq C^{\varphi\varphi^{-1}} \wedge D^{\varphi\varphi^{-1}} = C \wedge D$,

which gives $(C \wedge D)^{\varphi} \geq C^{\varphi} \wedge D^{\varphi}$. Hence we get $(C \wedge D)^{\varphi} = C^{\varphi} \wedge D^{\varphi}$.

∎

From Proposition 6.3.3., it follows that if A, B are in $SL(R)$ and $A \wedge B = \{1\}$, then we also have $A^{\varphi} \wedge B^{\varphi} = \{1\}$. If $M = A \times B$ is a subgroup of R with A, B in $SL(R)$, we wish to define $M^{\varphi} = A^{\varphi} \times B^{\varphi}$. We have to show that this definition is independent of the particular choice of A and B.

<u>Proposition 6.3.4.</u> Let A, B be elements of $SL(R)$, $A \cong B \cong C_{r^\alpha}$ and $A \wedge B = \{1\}$. Then we have $A \times B = A^{\varphi} \times B^{\varphi} = R_{\alpha}$.

<u>Proof</u> : From the hypothesis and Proposition 6.3.3., we have

$$AB = A \times B \cong C_{r^\alpha} \times C_{r^\alpha} \cong A^{\varphi} \times B^{\varphi}.$$

Hence $A \times B = A^{\varphi} \times B^{\varphi} = R_{\alpha}$ from the above Observation 6.3.2.. ∎

<u>Proposition 6.3.5.</u> Let A, B, C be elements of $SL(R)$ such that $A \cong C \cong C_{r^\alpha}$, $B \cong C_{r^\beta}$, $A \wedge B = \{1\}$ and $\alpha > \beta$. Then if $AC = A \times B$, we have

$$C^{\varphi} \leq A^{\varphi} \times B^{\varphi}.$$

<u>Proof</u> : From Proposition 6.3.3., we have $|A^{\varphi} \wedge C^{\varphi}| = |A \wedge C|$, and so

$$|A^{\varphi}C^{\varphi}| = |AC| = r^{\alpha+\beta}.$$

But then we must have $A^{\varphi}C^{\varphi} \cong C_{r^\alpha} \times C_{r^\beta}$, as $A^{\varphi}C^{\varphi}$ is contained in R_{α}, and so $A^{\varphi}C^{\varphi}$ contains R_{β}. Thus we get $B^{\varphi} \leq A^{\varphi}C^{\varphi}$ as $B^{\varphi} \leq R_{\beta}$. Therefore we have $A^{\varphi} \times B^{\varphi} \leq A^{\varphi}C^{\varphi}$, and we must have equality as they both have order $r^{\alpha+\beta}$. Hence $C^{\varphi} \leq A^{\varphi} \times B^{\varphi}$. ∎

<u>Proposition 6.3.6.</u> Let A, B, C be elements of $SL(R)$ such that $A \cong C_{r^\alpha}$, $B \cong C_{r^\beta}$, $A \wedge B = \{1\}$ and $\alpha \geq \beta$. Then if $C \leq A \times B$, we have

$$C^{\varphi} \leq A^{\varphi} \times B^{\varphi}.$$

<u>Proof</u> : If $\alpha = \beta$, we have $A \times B = R_{\alpha}$ and so from $C \leq R_{\alpha}$ we get

$$C^{\varphi} \leq R_{\alpha} = A^{\varphi} \times B^{\varphi}$$

by Proposition 6.3.4., and we are done. Now let

$X = \{(\alpha, \beta) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid 0 \leq \beta < \alpha,\ \exists\ A, B, C$ in $SL(R)$ such that

$A \cong C_r\alpha$, $B \cong C_r\beta$, $A \wedge B = \{1\}$, $C \leq A \times B$ and $C^\varphi \nleq A^\varphi \times B^\varphi\}$.
We want to show that X is empty . Suppose this is false and take $(\bar\alpha, \bar\beta)$ in X such that

$$(\alpha, \beta) \in X \Rightarrow \alpha \geq \bar\alpha \qquad \text{and} \qquad (\bar\alpha, \beta) \in X \Rightarrow \beta \geq \bar\beta .$$

Choose A , B , C in Γ such that $A \cong C_r\bar\alpha$, $B \cong C_r\bar\beta$, $A \wedge B = \{1\}$, $C \leq A \times B$ and $C^\varphi \nleq A^\varphi \times B^\varphi$. We shall find a final contradiction in a few steps :

(i). We must have $\bar\beta \geq 1$. For if $\bar\beta = 0$, then $B = \{1\}$ and so $C \leq A$ gives $C^\varphi \nleq A^\varphi$ which is a contradiction . Note that this also implies that $\bar\alpha \geq 2$.

(ii). We have $|C| = |A|$. Otherwise we would have $C \leq A^r \times B$. But then either $\bar\alpha = \bar\beta + 1$ and so $A^r \times B = R_{\bar\beta}$, which gives $C^\varphi \leq R_{\bar\beta} = (A^r)^\varphi \times B^\varphi \leq A^\varphi \times B^\varphi$, or $\bar\alpha > \bar\beta + 1$ and so we must have $C^\varphi \leq (A^r)^\varphi \times B^\varphi$ by minimality of $\bar\alpha$. But then we get $C^\varphi \leq A^\varphi \times B^\varphi$. In both cases we have a contradiction .

(iii). We have $AC = A \times B$. Suppose this is not the case , i.e. $AC < A \times B$. We can choose generators a , b , c for A , B , C resp. such that $c = ab^{r^\gamma}$ where $0 \leq \gamma \leq \bar\beta - 1$. So we get $|A \wedge C| = r^{\bar\alpha - \bar\beta + \gamma}$. But then , $AC < A \times B$ implies $\gamma \neq 0$, hence we have $C \leq A \times B^r$. Also , as $\bar\beta \geq 1$, we must have $C^\varphi \leq A^\varphi \times (B^r)^\varphi$, otherwise $(\bar\alpha, \bar\beta - 1)$ lies in X , which contradicts minimality of $\bar\beta$. Thus we obtain $C^\varphi \leq A^\varphi \times B^\varphi$, which gives a contradiction.

(iv). At this point we can apply Proposition 6.3.5. to get $C^\varphi \leq A^\varphi \times B^\varphi$, which gives the final contradiction. Hence X is empty and the result is proved. #

<u>Lemma 6.3.7.</u> Let A , A' , B , B' be elements of $SL(\mathrm{R})$ such that $A \cong A' \cong C_r\alpha$, $B \cong B' \cong C_r\beta$, $A \wedge B = A' \wedge B' = \{1\}$. Then we have

$A \times B = A' \times B'$ if and only if $A^\varphi \times B^\varphi = A'^\varphi \times B'^\varphi$.

<u>Proof</u> : Without loss of generality we may assume $\alpha \geq \beta$. Then the result follows from Proposition 6.3.6. applied to φ and φ^{-1} . #

We can now extend the definition of φ to all finite subgroups of R .

Let M be a finite subgroup of R. Then we have $M = A \times B$ for some A and B in $SL(R)$ such that $A \cong C_{p^\alpha}$, $B \cong C_{p^\beta}$ and $\alpha \geq \beta$. We define M^ϕ to be $A^\varphi \times B^\varphi$. Lemma 6.3.7. guarantees that this is a good definition.

<u>Proposition 6.3.8.</u> ϕ is inclusion preserving.

<u>Proof</u>: Let M, N be finite subgroups of R such that $M \leq N$. Choose A, B, C, D in $SL(R)$ such that $A \cong C_{p^\alpha}$, $B \cong C_{p^\beta}$, $C \cong C_{p^\gamma}$, $D \cong C_{p^\delta}$, $\alpha \geq \beta$, $\gamma \geq \delta$ and $M = A \times B$, $N = C \times D$. Then $M \leq N \Rightarrow A, B \leq C \times D$. So, by Proposition 6.3.6., we have $A^\varphi, B^\varphi \leq C^\varphi \times D^\varphi$. Hence we get

$$M^\phi = A^\varphi \times B^\varphi \leq C^\varphi \times D^\varphi = N^\phi,$$

and we are done. ∎

<u>Lemma 6.3.9.</u> Let X be a group. Suppose that for every n in $\mathbb{N}$ we have a subgroup X_n of X and an autoprojectivity φ_n of X_n such that

$X_n \leq X_{n+1}$ for every n in $\mathbb{N}$,

$\bigcup_{n=1}^{\infty} X_n = X$,

$Y^{\varphi_n} = Y^{\varphi_{n+1}}$ for every n in $\mathbb{N}$ and every subgroup Y of X_n.

Then there exists a unique autoprojectivity φ of X such that

$$Y^\varphi = Y^{\varphi_n}$$

for every n in $\mathbb{N}$ and every subgroup Y of X_n.

<u>Proof</u>: For uniqueness, suppose we have φ and ψ in $\operatorname{Aut} L(X)$ such that $Y^\varphi = Y^{\varphi_n} = Y^\psi$ for every n in $\mathbb{N}$ and every subgroup Y of X_n. Now suppose Y is a subgroup of X. If we put $Y_n = Y \wedge X_n$ for every n in $\mathbb{N}$, then we get

$$Y^\varphi = \bigcup_{n=1}^{\infty} Y_n^\varphi = \bigcup_{n=1}^{\infty} Y_n^{\varphi_n} = \bigcup_{n=1}^{\infty} Y_n^\psi = Y^\psi .$$

Hence $\varphi = \psi$. For the existence of φ, let Y be any subgroup of X. Define

$Y_n = Y \wedge X_n$ for every n in $\mathbb{N}$ and $Y^{\varphi} = \bigcup_{n=1}^{\infty} Y_n^{\varphi_n}$. From the hypothesis we get

$$Y_n^{\varphi_n} \le Y_n^{\varphi_{n+1}} \le Y_{n+1}^{\varphi_{n+1}}$$

and so Y^{φ} is actually a subgroup of X. We have therefore defined a map $\varphi: L(X) \to L(X)$. This map is inclusion preserving : for suppose Y and Z are subgroups of X such that $Y \le Z$. Then we have

$$Y_n = Y \wedge X_n \le Z \wedge X_n = Z_n$$

for every n in $\mathbb{N}$, and so

$$Y^{\varphi} = \bigcup_{n=1}^{\infty} Y_n^{\varphi_n} \le \bigcup_{n=1}^{\infty} Z_n^{\varphi_n} = Z^{\varphi}.$$

Now if we consider the family $\{\psi_n \mid \psi_n = \varphi_n^{-1}, n \in \mathbb{N}\}$, we still have $Y^{\psi_n} = Y^{\psi_{n+1}}$ for every n in $\mathbb{N}$ and every subgroup Y of X_n. So we may define an inclusion preseving map $\psi : L(X) \to L(X)$, in the same way we defined φ. If now Y is any subgroup of X we have

$$Y^{\varphi\psi} = \left(\bigcup_{n=1}^{\infty} Y_n^{\varphi_n}\right)^{\psi_n} = \bigcup_{n=1}^{\infty} Y_n^{\varphi_n\psi_n} = \bigcup_{n=1}^{\infty} Y_n = Y,$$

and similarly $Y^{\psi\varphi} = Y$. Hence we get $\varphi\psi = \psi\varphi = 1$ and φ is an autoprojectivity of X. #

We can now extend the definition of the map $\bar{\varphi}$, previously defined, to the whole of $L(R)$.

Consider the family $\{R_n \mid n \in \mathbb{N}\}$ of finite subgroups of R. Then we have

$R_n \le R_{n+1}$ for every n in $\mathbb{N}$ and

$\bigcup_{n=1}^{\infty} R_n = R$. Now we observe that with the same procedure we used to construct the inclusion preserving map $\bar{\varphi}$ on the set of all finite subgroups of R, we can construct an inclusion preserving map $\bar{\psi}$ starting from the bijection

φ^{-1} of $SL(R)$, and verify that $\phi\psi = \psi\phi = 1$. So , actually , ϕ is bijective and it induces an autoprojectivity of every finite subgroup of R which is fixed by ϕ . Therefore we can define autoprojectivities λ_n for every n in $\mathbb{N}$ by restricting ϕ to $L(R_n)$, as R_n is fixed by ϕ . At this point we can apply Lemma 6.3.9. to obtain an autoprojectivity ϕ of R . We can finally state

Lemma 6.3.10. If φ is an automorphism of the meet-semilattice $SL(R)$ of all cyclic subgroups of R , then there exists a unique autoprojectivity ϕ of R inducing φ on $SL(R)$.

Proof : We have already shown how to construct ϕ from φ . Uniqueness is then clear , because if M is any subgroup of R and θ is an autoprojectivity of R inducing φ on $SL(R)$, then we have

$$M^{\theta} = \bigvee_{x \in M} \langle x \rangle^{\theta} = \bigvee_{x \in M} \langle x \rangle^{\varphi} = M^{\phi} .$$

So $\theta = \phi$. ∎

We can now prove the following

Proposition 6.3.11. Let T be a 2-dimensional torus . If for every prime r , $r \neq p$, we have an automorphism φ_r of $SL(T_r)$, then there exists a unique autoprojectivity φ of T such that

$$X^{\varphi_r} = X^{\varphi}$$

for every prime r different from p and every subgroup X of T_r .

Proof : By Lemma 6.3.10. , we can extend every φ_r to an autoprojectivity ϕ_r of T_r . Then , if X is any subgroup of T , we define $X^{\varphi} = \bigvee X_r^{\phi_r}$ (r prime , $r \neq p$) . Then φ is an autoprojectivity of T and clearly we have $X^{\varphi_r} = X^{\varphi}$ for every prime r different from p and every subgroup X of T_r . Uniqueness follows from the fact that a projectivity is determined by its action on the meet- semilattice of cyclic subgroups . ∎

Remark : We actually have $\mathrm{Aut}\, L(T) \cong \mathrm{Cr}\, \mathrm{Aut}\, SL(T_r)$, as φ is index-preserving . (In general it is not true that

$$\mathcal{L}(G) \cong \mathrm{Cr}\, \mathcal{L}(G_\lambda) \Longrightarrow \mathrm{Aut}\, \mathcal{L}(G) \cong \mathrm{Cr}\, \mathrm{Aut}\, \mathcal{L}(G_\lambda) \, ,$$

as one can see by taking $G = K^{\times}$) .

We shall apply Proposition 6.3.11. to the case when T is the subgroup of diagonal matrices of $SL_3(\bar{F}_{23})$.

§ 6.4 The group $SL_3(\bar{F}_{23})$ has autoprojectivities which are not induced by any automorphism .

If G is a simple algebraic group of type A_2 , then G is isomorphic as an algebraic group either to $SL_3(K)$ or to $PGL_3(K)$. We shall show that for the groups $SL_3(K)$ the group Γ is not in general trivial , by showing that $\Gamma(SL_3(\bar{F}_{23}))$ contains a finite subgroup isomorphic to S_3 . To do this we shall first define a family of autoprojectivities of the group T of diagonal matrices of $SL_3(\bar{F}_{23})$.

From now on we shall denote by p the prime 23 , by K the field $\bar{F}_{23}$ and by G the group $SL_3(K)$ and by T the subgroup of diagonal matrices of G (so that T is a maximal torus of G) . We shall also denote by r the prime 11 .

We fix our attention on the r-component R of T . Then we have $R \cong C_{r^\infty} \times C_{r^\infty}$. The Weyl group $W = \mathcal{N}(T)/T$ acts on T and on $\mathcal{L}(T)$ in the usual way . We have

$R^w = R$ and $R_\alpha{}^w = R_\alpha$ for every w in W and every α in $\mathbb{N}$.

R_1 has 12 subgroups of order r , which are subdivided in three classes under the action of W on $\mathcal{L}(T)$. To give a description of these three classes , we fix an element ξ in $F_p^{\times}$ of order r and we let

$$a = \mathrm{diag}(\xi\, , \xi^{-1}, 1) \quad , \quad b = \mathrm{diag}(\xi\, , \xi\, , \xi^{-2}) \quad , \quad c = \mathrm{diag}(\xi\, , \xi^{2}, \xi^{-3}) \ .$$

Then , if we put $\mathcal{A} = \langle a \rangle^W$, $\mathcal{B} = \langle b \rangle^W$, $\mathcal{C} = \langle c \rangle^W$, where

$X^W = \{X^w \mid w \in W\}$ for every subgroup X of T,

we get that $\mathcal{A}$ and $\mathcal{B}$ have three elements, while $\mathcal{E}$ has six elements, i.e. W acts faithfully on $\mathcal{E}$ (this is because $\mathcal{N}(\langle e\rangle) = T$). Let $E = \langle e\rangle$.

We start now by constructing a family $(\varphi_w)_{w\in W}$ of autoprojectivities of R. Fix w in W. We define

$\{1\}^{\varphi_w} = \{1\}$ and E^{φ_w} by $E^{\varphi_w} = E^w$.

Also, if X is any cyclic subgroup of R containing E, we put $X^{\varphi_w} = X^w$.

Let then Y be any cyclic non-trivial subgroup of R. We define

$Y^{\varphi_w} = Y$ if $\Omega(Y)$ lies in $\mathcal{A}$ or $\mathcal{B}$.

Otherwise $\Omega(Y)$ must lie in $\mathcal{E}$, and so there exists a unique ρ in W such that $\Omega(Y) = E^{\rho}$. We then put

$Y^{\varphi_w} = Y^{\rho^{-1}w\rho}$.

So far, we have defined φ_w on all cyclic subgroups of R, i.e. on $\mathcal{SL}(R)$.

Proposition 6.4.1. Let w be in W. Then φ_w is inclusion preserving.

Proof: Let X, Y be cyclic subgroups of R such that $X \leq Y$. If $X = \{1\}$, there is nothing to prove. So let $X \neq \{1\}$. Then we must have $\Omega(X) = \Omega(Y)$. Hence, if $\Omega(X)$ lies in $\mathcal{A}$ or $\mathcal{B}$, we have $X^{\varphi_w} = X \leq Y = Y^{\varphi_w}$. Otherwise let ρ be the unique element of W for which $\Omega(X) = E^{\rho}$. Then we have

$X^{\varphi_w} = X^{\rho^{-1}w\rho} \leq Y^{\rho^{-1}w\rho} = Y^{\varphi_w}$,

and we are done. ∎

Proposition 6.4.2. Let w, w' be elements of W. Then we have

$\varphi_w \varphi_{w'} = \varphi_{w'w}$.

In particular φ_w is bijective for every w in W, with inverse $\varphi_{w^{-1}}$.

Proof: Let Y be a cyclic subgroup of R. We want to show that $Y^{\varphi_w\varphi_{w'}} = Y^{\varphi_{w'w}}$. If $Y = \{1\}$, the there is nothing to prove. So assume $Y \neq \{1\}$. If $\Omega(Y)$ lies in $\mathcal{A}$ or $\mathcal{B}$, we have

$Y^{\varphi_w} = Y^{\varphi_{w'}} = Y^{\varphi_{w'w}} = Y$, and so we are done . Otherwise let ρ be the unique element of W for which $\Omega(Y) = E^{\rho}$. Then we have $Y^{\varphi_w} = Y^{\rho^{-1}w\rho}$ and $Y^{\varphi_{w'w}} = Y^{\rho^{-1}w'w\rho}$. Also , $\Omega(Y^{\rho^{-1}w\rho}) = (\Omega(Y))^{\rho^{-1}w\rho} = E^{w\rho}$. Hence

$$Y^{\varphi_w\varphi_{w'}} = (Y^{\rho^{-1}w\rho})^{\varphi_{w'}} = Y^{\rho^{-1}w\rho(w\rho)^{-1}w'(w\rho)} = Y^{\rho^{-1}w'w\rho} = Y^{\varphi_{w'w}}$$

and we get the result . In particular , as from the definition it is clear that φ_1 is the identity , we have $\varphi_w \varphi_{w^{-1}} = \varphi_{w^{-1}} \varphi_w = \varphi_1 = 1$, and so φ_w is bijective with inverse $\varphi_{w^{-1}}$. ∎

Proposition 6.4.3. Let w be in W . Then φ_w commutes with the action of W on the cyclic subgroups of R .

Proof : Let Y be a cyclic subgroup of R , and σ be in W . We need to show that $(Y^{\sigma})^{\varphi_w} = (Y^{\varphi_w})^{\sigma}$. If $Y = \{1\}$, the there is nothing to prove . So assume $Y \neq \{1\}$. If $\Omega(Y)$ lies in $\mathcal{A}$ or $\mathcal{B}$, then the same is true for $\Omega(Y^{\sigma})$. Hence we have $(Y^{\sigma})^{\varphi_w} = Y^{\sigma}$ and $Y^{\varphi_w} = Y$, which leaves us with $(Y^{\sigma})^{\varphi_w} = Y^{\sigma} = (Y^{\varphi_w})^{\sigma}$, and so we are done . Otherwise let ρ be the unique element of W for which $\Omega(Y) = E^{\rho}$. Then we also have $\Omega(Y^{\sigma}) = E^{\rho\sigma}$. Thus , by definition, we get $Y^{\varphi_w} = Y^{\rho^{-1}w\rho}$ and $(Y^{\sigma})^{\varphi_w} = Y^{\sigma(\rho\sigma)^{-1}w\rho\sigma} = Y^{\rho^{-1}w\rho\sigma} = (Y^{\varphi_w})^{\sigma}$, and the result is proved . ∎

At this point , in view of Proposition 6.3.11. , we can extend the definition of φ_w to the whole of $L(T)$. As usual , if s is a prime , we denote by T_s the s-component of T . Then , for every prime s and for every w in W , we define the element $\psi_{s,w}$ of Aut $SL(T_s)$ as follows :

$\psi_{r,w} = \varphi_w$,

$\psi_{s,w} = 1$ if $s \neq r$.

Then , by Proposition 6.3.11. , we obtain a family $(\psi_w)_{w \in W}$ of autoprojectivities of T .

Proposition 6.4.4. Let w, w' be elements of W and let X be a subgroup of

T such that $X_T = \{1\}$. Then we have

$\psi_w \psi_{w'} = \psi_{w'w}$.

ψ_w commutes with the action of W on $L(T)$, and

ψ_w fixes X.

Proof : This comes from the way we constructed ψ_w and Proposition 6.4.2., 6.4.3.. #

Proposition 6.4.5. The map given by

$w \mapsto \psi_w$ for every w in W,

defines an antimonomorphism of W into $\mathrm{Aut}\, L(T)$.

Proof : From Proposition 6.4.4., we have that this map is an antihomomorphism, and it is clearly injective as

$w \neq 1$ implies $E^{\psi_w} = E^w \neq E$. #

Our aim is to show that it is possible to extend the definition of ψ_w to the whole of $L(G)$, for every w in W. For this purpose we introduce the groups $G_n = SL_3(p^{n!})$ for every n in $\mathbb{N}$, and the groups H_n of diagonal matrices of G. Then we shall have $G_n \leq G_{n+1}$ for every n in $\mathbb{N}$, and $\bigcup_{j \in \mathbb{N}} G_j = G$. We recall the notation we introduced in § 6.2. There we considered the finite field $\mathbb{F}_q$ and for every subfield F of $\mathbb{F}_q$, we defined $T(F)$ to be the group of diagonal matrices with entries in F. Here we just extend this definition by saying that for every subfield F of K, we defined $T(F)$ to be the subgroup of T whose entries are in F. Therefore we shall have $H_n = T(\mathbb{F}_{p^{n!}})$ for every n in $\mathbb{N}$. Furthermore, for every element $s = \mathrm{diag}(\lambda, \mu, \nu)$ of T, we define $\alpha_i(s)$ and $\mu_i(s)$, for $i = 1, 2, 3$, as we did in § 6.2. Also we recall Definition 6.2.1. for the definition of s to satisfy condition $(*)$ or $(**)$. If s satisfies $(*)$, then we still define $F(s)$ to be the subfield $\mathbb{F}_p(\alpha_1(s))$ $(= \mathbb{F}_p(\alpha_2(s)) = \mathbb{F}_p(\alpha_3(s)))$ of K. As we did in § 6.2, we shall write α_i, μ_i, and F instead of $\alpha_i(s)$, $\mu_i(s)$ and $F(s)$ if there is no ambiguity.

Observation 6.4.6. We observe that if X is a cyclic subgroup of T, and there exists a generator s of X satisfying $(*)$, then every generator of X satisfies $(*)$. Also, if there exists a generator s of X satisfying $(**)$, then every generator of X satisfies $(**)$. ∎

We shall use the properties of the autoprojectivities ψ_w to define certain autoprojectivities of the groups G_n, by using Proposition 6.2.9..

Proposition 6.4.7. Let ψ be an autoprojectivity of T. Then for every α in $\mathbb{N}$, and every subfield F of K, we have

$R_\alpha^\psi = R_\alpha$ and $T(F)^\psi = T(F)$.

Proof : Let α be in $\mathbb{N}$. We have $R^\psi = R$, because ψ is index-preserving. Hence we have $R_\alpha^\psi = R_\alpha$ as every autoprojectivity of R fixes R_α (being generated by all the cyclic subgroups of R of order p^α). Let now F be a subfield of K. Assume first that F is finite. Then $T(F)$ is the subgroup of T generated by all cyclic subgroups of T of order p^n-1, and therefore it is fixed by ψ. In general, F will be the union of the family $(F_a)_{a\in A}$ of its finite subfields F_a's. Hence we get $T(F) = \bigvee T(F_a)$ (for all $a \in A$), and so $T(F)^\psi = \bigvee T(F_a)^\psi = \bigvee T(F_a) = T(F)$. ∎

We need an explicit description of certain elements of R.
For every i in $\{0, 1, ..., r-1\}$, we define

$e_i = \mathrm{diag}(\xi, \xi^i, \xi^{-1-i})$ (so that $e = e_2$)

(recall that ξ is a fixed element of $F_p^{\times}$ of order r).

From a direct calculation it is possible to see that $\langle e_i \rangle$ lies in $\mathcal{E}$ if and only if i lies in the set $\{2, 3, 4, 6, 7, 8\}$.

Proposition 6.4.8. Suppose i lies in $\{0, 1, ..., r-1\}$ and $i \neq 0, 1, (r-1)/2$. Then we have

$\alpha_m(e_i) \neq \alpha_n(e_i)$ for every $m, n = 1, 2, 3$, $m \neq n$.

If also $i \neq r-2$, then we have

$|\alpha_m(e_i)| = r$ for every $m = 1, 2, 3$.

Proof : We have $\alpha_1(e_i) = \xi^{1-i}$, $\alpha_2(e_i) = \xi^{2i+1}$, $\alpha_3(e_i) = \xi^{2+i}$.

Then we get the result observing that the solutions in $\{0, 1, \ldots, r-1\}$ of the equations : $1-i \equiv 2i+1 \mod r$; $1-i \equiv 2+i \mod r$; $2i+1 \equiv 2+i \mod r$; $1-i \equiv 0 \mod r$; $2i+1 \equiv 0 \mod r$; $2+i \equiv 0 \mod r$ are respectively $i = 0$, $(r-1)/2$, 1, 1, $(r-1)/2$ and $r-2$. ∎

The next proposition is the crucial step in our construction.

Proposition 6.4.9. Let X be a cyclic r-subgroup of T such that $\Omega(X)$ is in $\mathcal{E}$. Then every generator of X satisfies (**).

Proof : There exists a unique i in $\{2, 3, 4, 6, 7, 8\}$ such that $\Omega(X) = \langle e_i \rangle$. Let r^a be the order of X, and choose η in K^* such that $\eta^{r^{a-1}} = \xi$. Then there exists a in $\mathbb{Z}$ such that $X = \langle \mathrm{diag}(\eta, \eta^a, \eta^{-1-a}) \rangle$. Call s this element $\mathrm{diag}(\eta, \eta^a, \eta^{-1-a})$. We must have $a \equiv i \mod r$, as $s^{r^{a-1}} = e_i$.

Now $\alpha_j(s)^{r^{a-1}} = \alpha_j(s^{r^{a-1}}) = \alpha_j(e_i)$, and so $\alpha_j(s)$ is a root of the polynomial $X^{r^{a-1}} - \alpha_j(e_i)$, which is in $\mathbb{F}_p[X]$. Hence

$\mu_j(s) \mid X^{r^{a-1}} - \alpha_j(e_i)$ for every $j = 1, 2, 3$.

But, from Proposition 6.4.8. we have $\alpha_j(e_i) \neq \alpha_k(e_i)$ for every $j, k = 1, 2, 3$, $j \neq k$, and so we get $\mu_j(s) \neq \mu_k(s)$ for every $j, k = 1, 2, 3$, $j \neq k$.

We now prove that $|\alpha_j(s)| = |s|$ for every $j = 1, 2, 3$, and $\deg \mu_j(s) = \deg \mu_k(s)$ for every $j, k = 1, 2, 3$. From Proposition 6.4.8., we have

$|\alpha_j(e_i)| = r$ for every $j = 1, 2, 3$,

and so we get

$|\alpha_j(s)| = r^a$ for every $j = 1, 2, 3$,

as $\alpha_j(s)^{r^{a-1}} = \alpha_j(e_i)$. Hence we get $|\alpha_j(s)| = |s|$ for every $j = 1, 2, 3$. But $|\alpha_j(s)| = r^a$ for every $j = 1, 2, 3$, also implies that $\mathbb{F}_p(\alpha_j(s)) = \mathbb{F}_p(\eta)$ for every $j = 1, 2, 3$, which means that $\deg \mu_j(s) = \deg \mu_k(s)$ for every $j, k = 1, 2, 3$. Therefore s satisfies (*). Now let n be any element of $\mathcal{N}(T)$. Then the element nsn^{-1} is a generator of the subgroup nXn^{-1} of T. Let us denote by X' this subgroup of T. But then $\Omega(X')$ lies in $\mathcal{E}$, and so, by the previous discussion, there exists a generator s' of X' satisfying $(*)$. But then, by Observation 6.4.6., every generator of X' satisfies $(*)$. Therefore nsn^{-1} satisfies $(*)$ and s satisfies $(**)$. But then every generator of X satisfies $(**)$ by Observation 6.4.6.. #

We need now to know the behaviour of finite subgroups of T not fixed by any autoprojectivity ψ_w, under the action of W.

Proposition 6.4.10. Let X be a finite subgroup of T. Suppose there exists w in W such that $X^{\psi_w} \neq X$. Then we have

$$X^{\sigma} \neq X$$

for every non-trivial element σ of W.

Proof: We can write $X = X_1 \times X_2$, where X_1 is contained in T_r and $(X_2)_r = \{1\}$. Then we have $X_2^{\psi_w} = X_2$, and so we must have $X_1^{\psi_w} \neq X_1$. Now $X_1 \leq T_r \Rightarrow X_1 = A \times B$, where A, B are cyclic subgroups of T_r such that $A \cong C_{\alpha}$, $B \cong C_{\beta}$ and $\alpha \geq \beta$. We must have $\alpha > \beta$, otherwise we get $X_1 = R_{\alpha}$ and so $X_1^{\psi_w} = X_1$, by Proposition 6.4.7.. Also we have $A^{\psi_w} \neq A$; suppose on the contrary that $A^{\psi_w} = A$. Then $X_1 = A \times B \geq R_{\beta} \geq B \Rightarrow X_1 = <A, R_{\beta}>$. But then $X_1^{\psi_w} = <A^{\psi_w}, R_{\beta}^{\psi_w}> = <A, R_{\beta}> = X_1$, which is a contradiction. So we have $A^{\psi_w} \neq A$, and this implies that $\Omega(A) \in \mathcal{E}$. Now let σ be in W, $\sigma \neq 1$. Then we have $A^{\sigma} \wedge A = \{1\}$, as $\Omega(A)^{\sigma} \neq \Omega(A)$. Hence we have $A^{\sigma} \nleq A \times B$, as $\beta < \alpha$, and so $X_1^{\sigma} \neq X_1$. But then $X^{\sigma} = X_1^{\sigma} \times X_2^{\sigma} \neq X_1 \times X_2 = X$, as we wanted. #

We can now start the last part of our construction. At the moment, we have a family $(\psi_w)_{w \in W}$ of autoprojectivities of T such that:

ψ_w commutes with the action of W on $L(T)$, for every w in W (Proposition 6.4.4.);

if X is a subgroup of T whose r-component is the identity, then $X^{\psi_w} = X$ for every w in W. In particular ψ_w fixes all the 2-subgroups and all the 3-subgroups of T (Proposition 6.4.4.);

the map given by $w \to \psi_w$ is an antimonomorphism of W into Aut $L(T)$ (Proposition 6.4.5.);

$R_\alpha{}^{\psi_w} = R_\alpha$ for every α in $\mathbb{N}$ and every w in W (Proposition 6.4.7.);

$T(F)^{\psi_w} = T(F)$ for every subfield F of K and every w in W (Proposition 6.4.7.);

ψ_w fixes every finite subgroup of T which is fixed by a non-trivial element of W, for every w in W (Proposition 6.4.10.).

We recall that for every n in $\mathbb{N}$ we have $H_n = T(F_{p^{n!}})$. Hence we have $H_n{}^{\psi_w} = H_n$ for every w in W and every n in $\mathbb{N}$. So we can define $\lambda_{w,n}$ to be the autoprojectivity of H_n obtained by restricting ψ_w to $L(H_n)$. From the properties of ψ_w, it follows that $\lambda_{w,n}$ satisfies conditions (i).- (iii). of Proposition 6.2.9.. We shall show that $\lambda_{w,n}$ also satisfies condition (iv')..

<u>Proposition 6.4.11.</u> Let s be a prime power order element of T not satisfying (**). Then we have

$$< s >^{\psi_w} = < s >$$

for every w in W.

<u>Proof</u>: Suppose, for a contradiction, that we have $< s >^{\psi_w} \neq < s >$ for some w in W. Then we must have $|s| = r^n$ for some n in $\mathbb{N}$, as ψ_w fixes every subgroup of T generated by an r'-element. Furthermore, $\Omega(< s >)$ must lie in $\mathcal{E}$, otherwise we would have $< s >^{\psi_w} = < s >$ by construction of ψ_w. But then $< s >$ is a cyclic r-subgroup of T such that $\Omega(< s >)$ lies in $\mathcal{E}$, and so, by Proposition 6.4.9., s satisfies (**), which contradicts the hypothesis that s does not satisfy (**). ∎

We can then summarize that for every w in W, for every n in $\mathbb{N}$, we have

(i). $\lambda_{w,n}$ commutes with the action of $\mathcal{N}_{G_n}(H_n)/H_n$ (Proposition 6.4.4. as this group acts on $L(H_n)$ in the same way as W does) ;

(ii). $\lambda_{w,n}$ fixes every subgroup of H_n which is fixed by a non-trivial element of $\mathcal{N}_{G_n}(H_n)/H_n$ (Proposition 6.4.10.);

(iii). $\lambda_{w,n}$ fixes every 2-subgroup and every 3-subgroup of H_n (by construction of $\lambda_{w,n}$) ;

(iv'). if s is a prime power order element of T not satisfying (**), then $\lambda_{w,n}$ fixes the subgroup generated by s (Proposition 6.4.11.).

We can therefore apply Proposition 6.2.9. to get the following

Proposition 6.4.12. For every w in W and for every n in $\mathbb{N}$, there exists a unique autoprojectivity $\theta_{w,n}$ of G_n fixing every p-subgroup of G_n and commuting with the inner automorphisms of G_n, such that

$$X^{\theta_{w,n}} = X^{\lambda_{w,n}}$$

for every subgroup X of H_n.

Proof : Existence follows from Proposition 6.2.9. , and uniqueness from the corollary on page 216 in [V_2]. #

We finally have

Proposition 6.4.13. For every w in W there exists a unique autoprojectivity δ_w of G, such that δ_w restricted to $L(G_n)$ is $\theta_{w,n}$, for every n in $\mathbb{N}$.

Proof : Let w be in W. Let n be in $\mathbb{N}$ and let X be a subgroup of G_n. Then we have $X^{\theta_{w,n}} = X^{\theta_{w,n+1}}$ because of uniqueness of $\theta_{w,n}$ (Proposition 6.4.12.). Then the result follows from Lemma 6.3.9. . #

From the construction it follows that δ_w fixes all unipotent subgroups of G and commutes with the inner automorphisms of G. We can therefore prove

Proposition 6.4.14. For every w in W, δ_w fixes all maximal tori and all

Borel subgroups of G. Hence δ_w lies in Γ.

Proof : Let T' be a maximal torus of G. Then there exists g in G such that $T' = T^g$. Hence we have

$$T'^{\delta_w} = (T^g)^{\delta_w} = (T^{\delta_w})^g = T^g = T'.$$

Now let B be a Borel subgroup of G and let T' be a maximal torus of B. Then we have

$$B^{\delta_w} = \langle T', R_u(B) \rangle^{\delta_w} = \langle T'^{\delta_w}, R_u(B)^{\delta_w} \rangle = \langle T', R_u(B) \rangle = B,$$

and we are done . ∎

Finally we can say we constructed an injective antihomomorphism of W into Γ, given by $w \to \delta_w$. In particular we have

Corollary 6.4.15. The group $SL_3(\bar{F}_{23})$ has autoprojectivites which are not induced by automorphisms .

Proof : We know that this is equivalent to the fact that Γ is not trivial . We showed that in our case Γ contains a subgroup which is isomorphic to S_3 (Proposition 6.4.5.). In particular it is not trivial . ∎

From the construction (but it had to be so) , we have $Z^{\delta_w} = Z$ for every w in W, where Z is the center of $SL_3(\bar{F}_{23})$. Hence also the group

$$PSL_3(\bar{F}_{23}) = SL_3(\bar{F}_{23})/Z$$

has autoprojectivities which are not induced by any automorphism . Therefore the same will be true for the group $PGL_3(\bar{F}_{23})$, which is isomorphic as an abstract group to $PSL_3(\bar{F}_{23})$.

Remark . In fact it is possible to show that $\Gamma(SL_3(\bar{F}_{23}))$ is infinite and even non-solvable .

Let us take a closer look at the group R (we recall that $R = T_{11} \cong C_{11^\infty} \times C_{11^\infty}$). We denote by E_1 (= E),..., E_6 the elements of $\mathcal{E}$. Then each is contained in exactly 11 cyclic subgroups of R of order 11^2. Let us denote these by

$E_{1,1}, \dots, E_{1,11}, E_{2,1}, \dots E_{2,11}, \dots, E_{6,1}, \dots, E_{6,11}$,

choosing the notation such that

$\Omega(E_{i,j}) = E_i$ for every $i = 1,\dots,6$, $j = 1,\dots,11$.

Each $E_{i,j}$, will be contained in 11 cyclic subgroups of R as well. We shall denote these by $E_{i,j,k}$, where $i \in \{1,\dots,6\}$ and $j, k \in \{1,\dots,11\}$, such that

$\Omega_2(E_{i,j,k}) = E_{i,j}$ for every $i = 1,\dots,6$, $j, k = 1,\dots,11$.

We can keep on with this procedure to obtain that every subgroup X of R such that $\Omega(X)$ lies in $\mathcal{E}$, has uniquely the form

$X = E_{i_1,i_2,\dots,i_n}$, where X has order p^n, $i_1 \in \{1,\dots,6\}$, $i_2,\dots,i_n \in \{1,\dots,11\}$, and

$\Omega_{n-1}(X) = E_{i_1,i_2,\dots,i_{n-1}}$ if $n > 1$.

Let Θ be the group $\{ (\sigma_i)_{i=2,\dots} \mid \sigma_i \in S_{11}$ for every $i = 2, 3, \dots \}$. We make Θ act on the set of all cyclic subgroups X of R such that $\Omega(X) = E_1$. Let X be a cyclic subgroup of R such that $\Omega(X) = E_1$. Then we have uniquely

$X = E_{1,i_2,\dots,i_n}$, where X has order p^n, and $i_2,\dots,i_n \in \{1,\dots,11\}$.

Let $\sigma = (\sigma_i)$ be in Θ. Then we define

$X^\sigma = E_{1,\sigma_2(i_2),\dots,\sigma_n(i_n)}$.

Suppose X, Y are subgroups of R such that $\Omega(X) = E_1$ and $X \le Y$. Then also $\Omega(Y) = E_1$, and we have uniquely

$X = E_{1,i_2,\dots,i_n}$, $Y = E_{1,j_2,\dots,j_m}$,

where X has order p^n, Y has order p^m, $i_2,\dots,i_n, j_2,\dots,j_m \in \{1,\dots,11\}$, and also $m \ge n$, $i_k = j_k$ for every $k = 2,\dots,n$, as $X \le Y$.

If now $\sigma = (\sigma_i)$ lies in Θ, we have

$X^\sigma = E_{1,\sigma_2(i_2),\dots,\sigma_n(i_n)}$, $Y^\sigma = E_{1,\sigma_2(j_2),\dots,\sigma_m(j_m)}$,

and so we get

$X^\sigma \le Y^\sigma$,

as $\sigma_k(i_k) = \sigma_k(j_k)$ for every $k = 2,\dots,n$.

It is possible , with the same argument we used in § 6.4 , to define an autoprojectivity φ_σ of T , for every σ in Θ . It will then turn out that $\Gamma(SL_3(\bar{F}_{23}))$ contains a subgroup isomorphic to Θ . In particular $\Gamma(SL_3(\bar{F}_{23}))$ is infinite and non-solvable .

We also add that it seems to be reasonable that we can extend to autoprojectivities of $SL_3(\bar{F}_{23})$ also automorphisms of $SL(T_s)$ for other primes s , and even that in general a similar procedure can be applied to the groups $SL_3(\bar{F}_p)$ for every prime p .

§ 6.5 Final remark .

In the previous chapter we proved that for simple algebraic groups of type A_2 , Theorem 3.4.9. does not hold . To prove this we showed that the group $\Gamma(SL_3(\bar{F}_{23}))$ contains a subgroup isomorphic to S_3 .

With the example we gave in the last paragraph , we actually proved that the case A_2 left out in the previous dicussion , represents indeed an exceptional case among the simple algebraic groups G over $\bar{F}_p$, from the subgroup lattice point of view .

Nevertheless we underline that we did not consider the case when p = 2 , with the exception of groups of rank 1 . We have some results also in this direction , even if we are still far from a complete understanding of the situation.

For instance , it is possible to extend the results obtained by Völklein ([V2]) for the groups $SL_n(2^a)/D$ (where D is any central subgroup of $SL_n(2^a)$) . Also, from the results we got in Chapter 4 , if the rank of G is at least 3 , then every exceptional autoprojectivity of G fixes every subgroup generated by semisimple elements of G .

The difficulties arise from the classification of unipotent conjugacy classes in characteristic 2 , which is usually much more complicated than the case of odd characteristic. We were able to prove that if φ lies in $\Gamma(G)$, where G is a simple algebraic group over $\overline{\mathbb{F}}_2$, then φ fixes all the subgroups of order 2 of G, using the classification of involutions ([Se]).

Finally I would like to mention the problem whether the projective image of a simple algebraic group G over $\overline{\mathbb{F}}_p$ is isomorphic (as an abstract group) to G. This is very likely to be true. It has been possible to prove this in the case when G is of adjoint type.

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